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Adaptive Auxiliary Loop for Output-Based Compensation of Perturbations in Linear Systems

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Abstract—The problem of output-feedback compensation of bounded additive perturbations affecting a minimum-phase linear system with unknown parameters is considered. An adaptive auxiliary loop is developed, which does not require to know the perturbation model and allows one to: *a*) separate the processes of estimation of parametric and additive perturbations, *b*) estimate and compensate for the additive perturbation with any given accuracy if the conditions of the parametric identifiability are met. The above-mentioned separated estimation of two disturbances of different nature is achieved by augmentation of the A.M. Tsykunov auxiliary loop method with the law to identify the unknown parameters, which is based on the instrumental variables approach and the procedure of dynamic regressor extension and mixing (DREM). The obtained system of the additive perturbations compensation has a certain potential to be used together with the conventional industrial PI-, PID-controllers. The theoretical results of this study are validated via mathematical modelling.

Keywords: disturbance, estimation, auxiliary loop, identification, instrumental variables, convergence, overparameterization, dynamic extension and mixing

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1. INTRODUCTION

The problem of external perturbation compensation has been attracting considerable attention of specialists in radio engineering, electrical engineering, control theory, etc. for many years. To date, two basic principles of such compensation have been proposed—indirect and direct ones.

Considering the indirect compensation, a characteristic polynomial of a closed-loop system is chosen so that the component of the system forced motion caused by the perturbation is reduced as much as possible in comparison with the one associated with the reference signal. However, it is impossible to ensure the same quality of compensation for perturbations with significantly different spectra by certain choice of the characteristic polynomial of a closed-loop system, and the problem of how to choose it on the basis of *a priori* data on the perturbation spectrum or in an optimal way (*e.g.*, in terms of $\mathcal{H}_2$-, $\mathcal{H}_\infty$-norms, invariant ellipsoid metrics, etc.) is faced. In this sense, it is necessary to admit the limitations of classical feedback.

A natural response to this challenge was the development of the direct compensation principle, according to which the control signal is decomposed into two components. The first one corrects the characteristic polynomial of the closed-loop system, while the second is to be equal to the perturbation with the opposite sign. If the disturbance is matched with the control signal and measurable, this approach achieves its full compensation. At first glance, the matching condition seems to be restrictive, because in practice the disturbance and the control signal are often unmatched. However, in fact, by following the equivalent-input-disturbance approach [1] and adopting some rather weak differentiability conditions for the original perturbation, the matching condition can always be satisfied. For example, consider the following system:

$$\begin{aligned}\dot{y} &= a_1 y + b_1 x + b_1 f,\\ \dot{x} &= a_2 x + b_2 u,\end{aligned}$$

then, if the perturbation f is differentiable, the application of the equivalent-input-disturbance approach allows one to obtain a system with matched perturbation:

$$\begin{aligned}&\dot{y} = a_1 y + b_1 \zeta,\\ &\dot{\zeta} = a_2 x + b_2 u + \dot{f} \pm a_2 f = a_2 \zeta + b_2 \left[u + \underbrace{b_2^{-1}\left(\dot{f} - a_2 f\right)}_{f_{eq}} \right],\end{aligned}$$

where $a_1,\ a_2,\ b_1,\ b_2$ are some scalars, y stands for a measurable output signal, x denotes unmeasurable state vector, $\zeta = x + f$ is a virtual state, u denotes a control signal, f stands for the original disturbance, f_{eq} is an equivalent input disturbance.

Thus, we will hereafter refer only to matched perturbations, assuming that the principle of equivalent-input-disturbance has already been implemented.

Much more restrictive requirement is that the perturbation is measurable. To relax it, various disturbance observers have been proposed in the literature that allow one to reconstruct the value of the disturbance with some (usually arbitrary) accuracy from measurements of the control and system output signals. Without pretending to provide an exhaustive review, some of the existing perturbation observers are considered further. We recommend an interested reader to study the reviews [2–5] to become aware of the full variety of methods.

Existing perturbation observers can be classified as follows:

1) methods that require to know both the system parameters and the perturbation model and parameters (extended Luenberger observer) [6],
2) methods that require knowledge of the system parameters and the disturbance model with parametric uncertainty [7–11],
3) methods that require the disturbance model to be known, while the system and disturbance parameters can be unknown [12–16],
4) methods requiring only knowledge of the system parameters [1, 17–20],
5*) methods that require neither knowledge of the disturbance model nor the system parameters [21–26].

The algorithms that belong to group 5* are the subject of interest of this study since, compared to other solutions, they require minimum amount of *a priori* information about the system and the perturbation. A detailed analysis of such algorithms shows that, in fact, there are no observers in the literature that allow one to estimate the additive perturbation if the system parameters are unknown. This is due to the fact that existing algorithms are not able to separate parametric and additive disturbances. Instead, the algorithms from the group 5* estimate an augmented perturbation consisting of their sum. Let us illustrate this statement with an example.

A first-order system is considered:

$$\dot{x} = u + \theta^{\top} \varphi(x) + f,$$

where $\theta^{\top}\varphi(x)$ is a parametric disturbance, f denotes an additive perturbation.

If the parameters θ are known and $\dot{f}$ is bounded, then a trivial observer ($s\colon = \frac{d}{dt}$):

$$\hat{f} = \frac{s}{ls+1}\left[x\right] - \theta^{\top}\frac{1}{ls+1}\left[\varphi\left(x\right)\right] - \frac{1}{ls+1}\left[u\right],$$

according to [26, p. 196; 27], allows one to estimate and compensate for the additive disturbance f with an arbitrary accuracy.

If the parameters θ are unknown, then only augmented disturbance can be estimated:

$$\frac{s}{ls+1}\left[x\right] - \frac{1}{ls+1}\left[u\right] = \frac{1}{ls+1}\left[f\right] + \theta^{\top}\frac{1}{ls+1}\left[\varphi\left(x\right)\right].$$

In order to single out an additive disturbance from the augmented one, the estimate of the parameters θ is to be obtained and used:

$$\hat{f} = \frac{s}{ls+1}\left[x\right] - \frac{1}{ls+1}\left[u\right] - \hat{\theta}^{\top}\frac{1}{ls+1}\left[\varphi\left(x\right)\right] = \frac{1}{ls+1}\left[f\right] - \tilde{\theta}^{\top}\frac{1}{ls+1}\left[\varphi\left(x\right)\right],$$

from which it follows that the separation of two types of perturbations is possible if and only if the parametric error $\tilde{\theta}$ converges to zero asymptotically. However, in case of perturbations, the existing identification laws provide only the parametric error boundedness [28, p. 556], which does not allow one to achieve complete separation even by augmenting the disturbance observer with known parameter estimation algorithms. Incomplete attempts to estimate additive and parametric uncertainties apart from each other can be found in [24, 29, 30].

The considered separation problem is of a fundamental nature and does not depend on the specific type of applied perturbation observer. Therefore, the existing observers estimate and compensate for an augmented perturbation represented as a sum of parametric and additive disturbances. Such an approach certainly deserves the right to exist and proved itself in comparison with the conventional PI- and PID-controllers [22] a long time ago. However, firstly, in such control systems the synergetic principle of least action [31] is violated, since the whole dynamics of the system is compensated without any reflection on its "usefulness" or "armfulness" to achieve the control objective; secondly, considering some practical scenarios, the baseline stabilising component of the control law (*e.g.*, PI- or PID-controller) has already been chosen by robust methods taking into account the parametric uncertainty ($\theta^{\top}\varphi\left(x\right)$ in the above-given example), and it is necessary to estimate and compensate only for the additive perturbation (f in the mentioned example). Therefore, the problem of design of the additive perturbation observer in the presence of parametric disturbance is actual.

In this study we consider the problem of estimation and output-based compensation of bounded additive perturbations affecting a minimum-phase linear system with unknown parameters. The solution of this problem is proposed to be obtained on the basis of the indirect adaptive control framework, according to which the control design procedure is decomposed into two stages. At the first one, a control law is introduced that ensures the achievement of the control objective assuming that the system parameters are known (in this study, we use the method of A.M. Tsykunov auxiliary loop [26, p. 196] to design such a law). At the second stage, the identification law is developed, and all unknown parameters of the control law defined at the first stage are substituted with their dynamic estimates. The key structural element of such an adaptive control system is the above-mentioned law, which is required to ensure asymptotic convergence of the unknown parameter estimates to their true values in a closed loop affected by an additive disturbance under the weakest possible regressor excitation requirements. In this paper, a recently proposed algorithm [32] based on the instrumental variables method [33] and the dynamic regressor extension and mixing (DREM) procedure [34] is proposed to solve the online identification problem. The convergence conditions of the parameter identification process using such a law are given below:

—a controller that stabilises the system affected by the additive and parametric perturbations is known,

—a control signal to compensate for the disturbance is bounded (for instance, via sat $\{.\}$ function),

—a reference signal includes not less than n different frequencies (where n is the system order),

—the reference and disturbance signals spectra have no common frequencies.

It is shown that when these conditions are met, the parametric error convergences to zero despite the presence of the additive disturbance, and the proposed system of adaptive compensation of such perturbation ensures its asymptotic estimation and compensation. The potential of application of the proposed system together with the conventional industrial PI-, PID-controllers is shown via numerical experiments, the fact that the conditions of parametric convergence are met when typical reference signals are used is also illustrated.

2. PROBLEM STATEMENT

The following perturbed linear dynamic system is considered:

$$\begin{aligned} y(t) &= \frac{Z(\theta,\ s)}{R(\theta,\ s)}\left[u(t)+f(t)\right], \\ Z(\theta,\ s) &= b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0, \\ R(\theta,\ s) &= s^n + a_{n-1}s^{n-1} + \ldots + a_0, \end{aligned} \tag{2.1}$$

where $y(t)$ is a measurable output signal, $u(t)$ stands for a control signal to be designed, $f(t)$ denotes an unknown bounded disturbance, $\theta \in D_\theta \subset \mathbb{R}^{n+m+1}$ is a vector of unknown parameters of an open-loop system, $R(\theta,\ s)$, $Z(\theta,\ s)$ are polynomials of order n and $m \leqslant n-1$, respectively, $s[.] := \frac{d}{dt}[.]$ denotes a differential operator.

Further, the control signal $u(t)$ is assumed to be chosen as:

$$\begin{aligned} u(t) &= u_b(t) + u_c(t), \\ u_b(t) &= \frac{P_y(\kappa,s)}{Q_y(\kappa,\ s)}[y(t)] + \frac{P_r(\kappa,s)}{Q_r(\kappa,\ s)}[r(t)] = \frac{P_y(\kappa,s)}{Q_y(\kappa,\ s)}[y(t)] + r_f(t), \end{aligned} \tag{2.2}$$

where $u_b(t)$ is a baseline component to stabilize the system, $u_c(t)$ stands for a summand to compensate for a disturbance $f(t)$, $\kappa \in D_\kappa \subset \mathbb{R}^{n_\kappa}$ denotes known time-invariant parameters of the control law, $r(t)$ is a reference signal, $m_y \leqslant n_y$ and $m_r \leqslant n_r$ are orders of pairs of polynomials $P_y(\kappa,\ s)$, $Q_y(\kappa,\ s)$ and $P_r(\kappa,\ s)$, $Q_r(\kappa,\ s)$, respectively.

For a rigorous formal problem statement, together with the system (2.1), its parametrization in the form of a linear regression equation is also considered:

$$z(t) = \varphi^\top(t)\,\theta + w(t), \tag{2.3}$$

where

$$z(t) = \frac{s^n}{\Lambda(s)}y(t),\ \varphi(t) = \begin{bmatrix} -\frac{\alpha_{n-1}^\top(s)}{\Lambda(s)}[y(t)] & \frac{\alpha_m^\top(s)}{\Lambda(s)}[u(t)] \end{bmatrix}^\top,$$

$$w(t) = \begin{bmatrix} b_m & b_{m-1} & \ldots & b_0 \end{bmatrix}\frac{\alpha_m(s)}{\Lambda(s)}[f(t)],$$

$$\alpha_{n-1}^\top(s) = \begin{bmatrix} s^{n-1} & \cdots & s & 1 \end{bmatrix},\ \alpha_m^\top(s) = \begin{bmatrix} s^m & \cdots & s & 1 \end{bmatrix},$$

$\varphi(t) \in \mathbb{R}^{m+n+1}$ is a measurable regressor, $z(t) \in \mathbb{R}$ stands for a measurable regressand, $\Lambda(s)$ denotes a stable polynomial of order n.

We adopt the following assumptions with respect to the stabilizing component of the control law and the disturbance.

Assumption 1. The transfer function of the closed-loop system (θ_{cl}: $\mathbb{R}^{n+m+1} \times \mathbb{R}^{n_\kappa} \mapsto \mathbb{R}^{n_{cl}+m_{cl}+1}$ are unknown parameters of the closed-loop system):

$$W_{cl}(\theta_{cl},\ s)[.] = \frac{Z(\theta,\ s)\, Q_y(\kappa,\ s)}{[Q_y(\kappa,\ s)\, R(\theta,\ s) - Z(\theta,\ s)\, P_y(\kappa,\ s)]}[.] = \frac{Z_{cl}(\theta_{cl},\ s)}{R_{cl}(\theta_{cl},\ s)}[.] \tag{2.4}$$

has Hurwitz polynomials $Z_{cl}(\theta_{cl},\ s)$ and $R_{cl}(\theta_{cl},\ s)$ for certain time-invariant κ from D_κ and any θ from D_θ.

Assumption 2. There exists a known function μ: $[t_0,\ \infty) \mapsto \mathbb{R}_{>}$ such that for

$$\lambda(t) = \frac{-1}{m_0 \mu^{n+1}(t)} \left(\frac{d^{n+1}}{dt^{n+1}}[f(t)] + \sum_{i=1}^{n} m_i \mu^{n-(i-1)}(t) \frac{d^i}{dt^i}[f(t)] \right)$$

for any $m_i,\ i = 0, \ldots,\ n$ it holds that $\mu\lambda \in L_\infty$ and $\lambda \in L_p \cap L_\infty$ for $p \in [1,\ \infty)$.

Assumption 3. An instrumental variable $\zeta(t) \in \mathbb{R}^{n+m+1}$, which is defined as follows (where $\theta_{iv} \in D_\theta \subset \mathbb{R}^{n+m+1}$ are known parameters of the instrumental variable):

$$\begin{aligned}
\zeta(t) &= \left[-\frac{\alpha_{n-1}^\top(s)}{\Lambda(s)}[y_{iv}(t)] \quad \frac{\alpha_m^\top(s)}{\Lambda(s)}[u_{iv}(t)] \right]^\top, \\
y_{iv}(t) &= \frac{Z(\theta_{iv}, s)}{R(\theta_{iv},\ s)}[u_{iv}(t)], \\
u_{iv}(t) &= \frac{P_y(\kappa, s)}{Q_y(\kappa,\ s)}[y_{iv}(t)] + \frac{P_r(\kappa,\ s)}{Q_r(\kappa,\ s)}[r(t)],
\end{aligned} \tag{2.5}$$

and a filtered disturbance $w(t)$ are independent, *i.e.*:

$$\forall t \geqslant t_0 \left| \int_{t_0}^{t} \zeta_i(s)\, w(s)\, ds \right| \leqslant c < \infty\ \forall i = 1, \ldots,\ n+m+1. \tag{2.6}$$

Based on closed-loop equation (2.4), the required behavior of the system is defined as:

$$y_{ref}(t) = W_{cl}(\theta_{cl},\ s)[r_f(t)]. \tag{2.7}$$

The aim is to design the compensation control signal $u_c(t)$, which does not include the output signal $y(t)$ derivatives, in such a way that the following equalities hold:

$$\overline{\lim_{t\to\infty}} |y(t) - y_{ref}(t)| = \overline{\lim_{t\to\infty}} |\tilde{y}(t)| = 0. \tag{2.8}$$

Therefore, the problem of estimation and compensation of an unknown perturbation that affects a minimum-phase system with unknown parameters is stated in this study.

Remark 1. Considering practical scenarios, we almost always know the stabilizing controller $u_b(t)$, which ensures the that the characteristic polynomial of the closed-loop system $R_{cl}(\theta_{cl},\ s)$ is Hurwitz one (*e.g.*, in some situations it can be chosen via trial and error). The polynomial $Z_{cl}(\theta_{cl}, s)$ to be Hurwitz requires the plant (2.1) to be minimum phase.

Remark 2. It should be noted that the output of the reference model (2.4) is not measurable, and the reference model itself defines different control quality for each specific parameter vector of the system θ from D_θ. These facts essentially distinguish the problem solved in this study from the classical problem of model reference adaptive control, in which the reference model is the same for all θ.

Remark 3. Assumption 2 restricts a class of admissible external disturbances. As for practical scenarios, there almost always exists $p \in [1,\ \infty)$ such that $\frac{d^i}{dt^i} f(t) \in L_p$ for all $i = 1, \ldots, n+1$, and, to meet such assumption, it is sufficient to choose $\mu(t) = \text{const} = \mu > 0$. If $\frac{d^i}{dt^i} f(t) \notin L_p$ for all $p \in [1,\ \infty)$, but $\sup\limits_t \left|\frac{d^i}{dt^i} f(t)\right| < \infty$, then it is sufficient to choose $\mu(t) = \mu_0 t + \mu_1$ with arbitrary $\mu_0 > 0$, $\mu_1 > 0$. Unfortunately, if the disturbance $f(t)$ has unbounded time derivatives, some additional *a priori* information is required for reasonable choice of $\mu(t)$.

Remark 4. In Assumption 3, the necessary conditions of asymptotic convergence of the identification law [32] are assumed to be met. In a stationary case, the inequality (2.6) is satisfied if the spectra of $r(t)$ and $f(t)$ do not have common frequencies.

3. MAIN RESULT

The description of the proposed method to solve the problem (2.8) is decomposed into three parts. In the first one, the filtered equivalent of the perturbation is parameterized as a function of the control and output signals and some unknown parameters calculated via θ. In the second part, an identification law for the unknown parameters is designed on the basis of the perturbation parametrization. In the third part, based on the obtained parametrization of the filtered perturbation and identification law, an adaptive signal for disturbance compensation is introduced and the achievement of the objective (2.8) is proved.

3.1. Disturbance Parametrization

Temporarily, the parameters θ are assumed to be known and, following [26], an auxiliary model is introduced:

$$\hat{y}^*(t) = \frac{Z(\theta,\ s)}{R(\Theta,\ s)}[u(t)], \tag{3.1}$$

where $\Theta \in \mathbb{R}^n$ are known parameters of the auxiliary model such that the system (3.1) is stable.

Owing to equations (3.1) and (2.1), the error $\varepsilon^*(t) = y(t) - \hat{y}^*(t)$ is written as follows:

$$\varepsilon^*(t) = \frac{Z(\theta,\ s)}{R(\Theta,\ s)}[f(t)] + \frac{R(\Theta,\ s) - R(\theta,\ s)}{R(\Theta,\ s)}[y(t)]. \tag{3.2}$$

Then, if $n - m$ derivatives of the signals $\varepsilon^*(t)$ and $y(t)$ are measurable, and the polynomial $Z(\theta, s)$ is Hurwitz one, then the following signal

$$u_c(t) = -\frac{R(\Theta,\ s)}{Z(\theta,\ s)}[\varepsilon^*(t)] + \frac{R(\Theta,\ s) - R(\theta,\ s)}{Z(\theta,\ s)}[y(t)] = -f(t) \tag{3.3}$$

ensures full compensation of the disturbance:

$$y(t) = \varepsilon^*(t) + \hat{y}^*(t) = \frac{Z(\theta,\ s)}{R(\theta,\ s)}[u_b(t)] = W_{cl}(\theta_{cl},\ s)[r_f(t)] = y_{ref}(t). \tag{3.4}$$

The perturbation and its compensator (3.3) can be represented in the following equivalent form:

$$f(t) = -u_c(t) = \psi_a^\top(\Theta)\,\alpha(s)\left[\varepsilon_f(t)\right] + \left(\psi_a^\top(\theta_a) - \psi_a^\top(\Theta)\right)\alpha(s)\left[y_f(t)\right], \tag{3.5a}$$

$$\begin{array}{ll} \dot{\xi}_\varepsilon(t) = A_b\xi_\varepsilon(t) + \rho e_m \varepsilon^*(t), & \dot{\xi}_y(t) = A_b\xi_y(t) + \rho e_m y(t), \\ \begin{cases} \varepsilon_f(t) = e_1^\top \xi_\varepsilon(t), \text{ if } m \geqslant 1 \\ \varepsilon_f(t) = \rho\varepsilon^*(t), \text{ if } m = 0, \end{cases} & \begin{cases} y_f(t) = e_1^\top \xi_y(t), \text{ if } m \geqslant 1 \\ y_f(t) = \rho y(t), \text{ if } m = 0, \end{cases} \end{array} \tag{3.5b}$$

$$\begin{cases} \dot{\hat{x}}^*(t) = A_0\hat{x}^*(t) - \Theta\hat{y}^*(t) + \theta_b u(t) \\ \hat{y}^*(t) = e_1^\top \hat{x}^*(t), \end{cases} \tag{3.5c}$$

where

$$A_b = A_0 - \psi_b(\theta)\,e_1^\top,\ \rho = \frac{1}{b_m} = \frac{1}{e_1^\top\theta_b},$$

$$\psi_b(\theta) = \begin{bmatrix} \frac{b_{m-1}}{b_m} & \frac{b_{m-2}}{b_m} & \cdots & \frac{b_0}{b_m} \end{bmatrix}^\top = \rho\mathcal{L}_\psi\theta_b,$$

$$\psi_a(\theta_a) = \begin{bmatrix} \mathcal{I}_n\theta_a & 1 \end{bmatrix}^\top,$$

$$\theta_a = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix}^\top = \mathcal{L}_a\theta,$$

$$\theta_b = \begin{bmatrix} b_m & b_{m-1} & \cdots & b_0 \end{bmatrix}^\top = \mathcal{L}_b\theta,$$

$$\mathcal{L}_a = \begin{bmatrix} I_{n\times n} & 0_{n\times(m+1)} \end{bmatrix},\ \mathcal{L}_b = \begin{bmatrix} 0_{(m+1)\times n} & I_{(m+1)\times(m+1)} \end{bmatrix},$$

$$\mathcal{L}_\psi = \begin{bmatrix} 0_{m\times 1} & I_{m\times m} \end{bmatrix},\ \alpha(s) = \begin{bmatrix} 1 & \cdots & s^{n-1} & s^n \end{bmatrix},$$

and $\mathcal{I}_n$ is a matrix, which secondary diagonal contains ones, while all other elements are zeros, e_i denotes a vector, which ith element is one, while all other ones are zeros, A_0 stands for an upper-shift matrix of respective dimension, $\theta_a,\ \theta_b$ are components of the vector $\theta = \begin{bmatrix} \theta_a^\top & \theta_b^\top \end{bmatrix}^\top$ and simultaneously parameters of the polynomials $R(\theta,\ s)$ and $Z(\theta,\ s)$.

According to the problem statement, the parameters θ are unknown and the derivatives of the signals $\varepsilon^*(t)$ and $y(t)$ are not directly measurable, which means that the compensator (3.3), (3.5a) can not be implemented. The requirement to know the derivatives of the mentioned signals can be relaxed by design of filtered derivative observers.

Statement 1. *Define:* 1) *the observers of the ith filtered derivative of the signals* $y_f(t)$ *and* $\varepsilon_f(t)$

$$\begin{cases} \dot{H}_i^\varepsilon(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)H_i^\varepsilon(t) + e_{n+1}\left(K_0(t)\,\varepsilon_f(t) - v_\varepsilon(t)\right) \\ h_i^\varepsilon(t) = e_{i+1}^\top H_i^\varepsilon(t),\ v_\varepsilon(t) = \sum\limits_{j=0}^{i} v_j^\varepsilon(t,\ M_0,\ K_0), \end{cases}$$
$$\begin{cases} \dot{H}_i^y(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)H_i^y(t) + e_{n+1}\left(K_0(t)\,y_f(t) - v_y(t)\right) \\ h_i^y(t) = e_{i+1}^\top H_i^y(t),\ v_y(t) = \sum\limits_{j=0}^{i} v_j^y(t,\ M_0,\ K_0), \end{cases} \quad \forall i = 0, \ldots, n,$$

where

$$M_0^\top(t) = \begin{bmatrix} -m_0\mu^{n+1}(t) & \ldots & -m_{n-1}\mu^2(t) & -m_n\mu(t) \end{bmatrix},$$

$$G_0 = \begin{bmatrix} 0_{n+1} & \begin{matrix} I_{n\times n} \\ 0_{1\times n} \end{matrix} \end{bmatrix},\ K_0(t) = m_0\mu^{n+1}(t),$$

and

$$\begin{cases} v_0^{\varepsilon}\left(t,\ M_0,\ K_0\right) = 0 \\ v_1^{\varepsilon}\left(t,\ M_0,\ K_0\right) = s^{-1}\left[s\left[M_0^{\top}\left(t\right)\right]H_i^{\varepsilon}\left(t\right)\right] + s^{-1}\left[s\left[K_0\left(t\right)\right]\varepsilon_f\left(t\right)\right] \\ \vdots \\ v_j^{\varepsilon}\left(t,\ M_0,\ K_0\right) = v_{j-1}^{\varepsilon}\left(t,\ M_0,\ K_0\right) - s^{-1}\left[v_{j-1}^{\varepsilon}\left(t,\ \dot{M}_0,\ \dot{K}_0\right)\right], \quad j = 2, \ldots i, \end{cases}$$

$$\begin{cases} v_0^{y}\left(t,\ M_0,\ K_0\right) = 0 \\ v_1^{y}\left(t,\ M_0,\ K_0\right) = s^{-1}\left[s\left[M_0^{\top}\left(t\right)\right]H_i^{y}\left(t\right)\right] + s^{-1}\left[s\left[K_0\left(t\right)\right]y_f\left(t\right)\right] \\ \vdots \\ v_j^{y}\left(t,\ M_0,\ K_0\right) = v_{j-1}^{y}\left(t,\ M_0,\ K_0\right) - s^{-1}\left[v_{j-1}^{y}\left(t,\ \dot{M}_0,\ \dot{K}_0\right)\right], \quad j = 2, \ldots i, \end{cases}$$

and scalars $m_0,\ m_1,\ \ldots, m_n$ *are coefficients of a stable polynomial;*

2) *the filtered disturbance:*

$$\begin{gathered} f_f\left(t\right) := \psi_a^{\top}\left(\Theta\right)h_{\varepsilon}\left(t\right) + \left(\psi_a^{\top}\left(\theta_a\right) - \psi_a^{\top}\left(\Theta\right)\right)h_y\left(t\right), \\ h_y\left(t\right) = \begin{bmatrix} h_0^y\left(t\right) & \cdots & h_i^y\left(t\right) & \cdots & h_n^y\left(t\right)\end{bmatrix}^{\top}, \\ h_{\varepsilon}\left(t\right) = \begin{bmatrix} h_0^{\varepsilon}\left(t\right) & \cdots & h_i^{\varepsilon}\left(t\right) & \cdots & h_n^{\varepsilon}\left(t\right)\end{bmatrix}^{\top}. \end{gathered} \tag{3.6}$$

Then, if Assumptions 1 and 2 are met, then:

1) *the error* $\tilde{f}\left(t\right) = f_f\left(t\right) - f\left(t\right)$ *converges asymptotically to zero* $\lim\limits_{t\to\infty}\tilde{f}\left(t\right) = 0$,

2) $\tilde{f} \in L_p \cap L_{\infty}$ *for* $p \in [1,\ \infty)$.

Proof of Proposition 1 is postponed to Appendix.

Note that in the stationary case $\mu\left(t\right) = \text{const}$, the filtered derivative observers proposed in Proposition 1 are reduced to proper differentiators. According to Proposition 1, if the parameters θ are known, then instead of the true perturbation $f\left(t\right)$, using measurable signals only, it is possible to calculate some filtered perturbation $f_f\left(t\right)$, which asymptotically converges to the true perturbation if Assumptions 1 and 2 are met. In this case, the signal

$$u_c\left(t\right) = -f_f\left(t\right) \tag{3.7}$$

allows one to obtain the following result.

Theorem 1. *If the parameters* θ *are known, and Assumptions 1 and 2 are met, then the control law* (2.2) + (3.7) *ensures* $\tilde{y} \in L_{\infty}$ *and* $\lim\limits_{t\to\infty}\left|\tilde{y}\left(t\right)\right| = 0$.

Proof of Theorem 1 is given in Appendix.

The requirement to know the parameters θ is relaxed by application of the identification law proposed in [32] and design of an adaptive auxiliary model (3.1) and an adaptive version of the compensation signal (3.7).

3.2. Unknown Parameters Identification

First of all, in order to implement the adaptive compensation signal, the estimates $\hat{\theta}(t)$, $\hat{\rho}(t)$, $\hat{\psi}_b(t)$ of the system unknown parameters and the disturbance parametrization (3.6) are required to be obtained. It should be noted that we do not need to identify the parameters $\psi_a\left(\theta_a\right)$, as the function $\psi_a\colon \mathbb{R}^{n+m+1} \mapsto \mathbb{R}^{n+m+2}$ obviously satisfies the Lipschitz condition, and, consequently, the estimate $\hat{\psi}_a\left(t\right)$ can be obtained via direct substitution $\hat{\psi}_a\left(t\right) := \psi_a\left(\hat{\theta}_a\right)$, where $\hat{\theta}_a\left(t\right) = \mathcal{L}_a\hat{\theta}\left(t\right)$. To make the effect of the adaptive law of perturbation compensation, which is based on dynamic estimates, asymptotically equivalent to the effect of the ideal compensation signal (3.1), (3.7), at least asymptotic convergence of parametric errors to zero is necessary to be ensured. For this purpose, the identification law developed in [32] will be used.

Applying the instrumental variable (2.5), the regression equation (2.3) is extended by means of averaging and sliding window filters:

$$\begin{aligned}&\dot{\vartheta}(t)=\zeta_{iv}(t)\,z(t)-\zeta_{iv}(t-T)\,z(t-T),\ \vartheta(t_0)=0_{n+m+1},\\&\dot{\psi}(t)=\zeta_{iv}(t)\,\varphi^\top(t)-\zeta_{iv}(t-T)\,\varphi^\top(t-T),\ \psi(t_0)=0_{(n+m+1)\times(n+m+1)},\end{aligned}\tag{3.8}$$

$$\begin{aligned}&\dot{Y}(t)=-\frac{1}{F(t)}\dot{F}(t)\,(Y(t)-\vartheta(t)),\ Y(t_0)=0_{n+m+1},\\&\dot{\Phi}(t)=-\frac{1}{F(t)}\dot{F}(t)\,(\Phi(t)-\psi(t)),\ \Phi(t_0)=0_{(n+m+1)\times(n+m+1)},\\&\dot{F}(t)=pt^{p-1},\ F(t_0)=F_0>0,\end{aligned}\tag{3.9}$$

where $T>0$ denotes a sliding window width, $p\geqslant 1,\ F_0\geqslant t_0^p$ stand for the filter parameters.

Application of filtration (3.8) and (3.9) in case $\theta=\text{const}$ allows one to obtain the following regression equation [32]:

$$Y(t)=\Phi(t)\,\theta+W(t),\tag{3.10}$$

where the disturbance $W(t)$ satisfies the following equations:

$$\begin{aligned}&\dot{W}(t)=-\frac{1}{F(t)}\dot{F}(t)\,(W(t)-\varepsilon(t)),\ W(t_0)=0_{n+m+1},\\&\dot{\varepsilon}(t)=\zeta_{iv}(t)\,w(t)-\zeta_{iv}(t-T)\,w(t-T),\ \varepsilon(t_0)=0_{n+m+1}.\end{aligned}\tag{3.11}$$

Having multiplied (3.10) by $\operatorname{adj}\{\Phi(t)\}$, the set of scalar regression equations is obtained:

$$\begin{gathered}\mathcal{Y}(t)=\Delta(t)\,\theta+\mathcal{W}(t),\\\mathcal{Y}(t):=\operatorname{adj}\{\Phi(t)\}\,Y(t),\quad\Delta(t):=\det\{\Phi(t)\},\quad\mathcal{W}(t):=\operatorname{adj}\{\Phi(t)\}\,W(t).\end{gathered}\tag{3.12}$$

Based on the regression equation (3.10), the estimate of the parameters θ can be obtained via application of the gradient descent method. However, in order to implement the compensation component (3.1), (3.7), the estimates of the parameters $\psi_b(\theta)$ and ρ are required. It should be noted that, owing to the equality $\psi_b(\theta)=\rho\mathcal{L}_\psi\mathcal{L}_b\theta$, it is sufficient to have estimates of the parameters ρ and θ to obtain $\hat{\psi}_b(t)$. The value of $\hat{\theta}(t)$ can be calculated via (3.12), and therefore, now we need to derive a regression equation with respect to ρ.

Having multiplied (3.12) by $e_1^\top\mathcal{L}_b$, we have

$$\begin{aligned}&e_1^\top\mathcal{L}_b\mathcal{Y}(t)=\Delta(t)\,e_1^\top\mathcal{L}_b\theta+e_1^\top\mathcal{L}_b\mathcal{W}(t)\\&\Rightarrow e_1^\top\mathcal{L}_b\mathcal{Y}(t)=\Delta(t)\,b_m+e_1^\top\mathcal{L}_b\mathcal{W}(t),\end{aligned}$$

from which the required regression equation is obtained via multiplication by ρ

$$\begin{gathered}\mathcal{Y}_\rho(t)=\mathcal{M}_\rho(t)\,\rho+\mathcal{W}_\rho(t),\\\mathcal{Y}_\rho(t):=\Delta(t),\quad\mathcal{M}_\rho(t):=e_1^\top\mathcal{L}_b\mathcal{Y}(t),\quad\mathcal{W}_\rho(t)=-\rho e_1^\top\mathcal{L}_b\mathcal{W}(t).\end{gathered}\tag{3.13}$$

Based on the regression equations (3.12) and (3.13), the laws are designed to estimate all parameters that are necessary for the implementation of (3.1), (3.7):

$$\begin{gathered}\dot{\hat{\theta}}(t)=-\gamma\Delta(t)\left(\Delta(t)\,\hat{\theta}(t)-\mathcal{Y}(t)\right),\ \hat{\theta}(t_0)=\hat{\theta}_0,\\\dot{\hat{\rho}}(t)=-\gamma_\rho\mathcal{M}_\rho(t)\,(\mathcal{M}_\rho(t)\,\hat{\rho}(t)-\mathcal{Y}_\rho(t)),\ \hat{\rho}(t_0)=\hat{\rho}_0,\\\hat{\psi}_b(t)=\hat{\rho}(t)\,\mathcal{L}_\psi\mathcal{L}_b\hat{\theta}(t).\end{gathered}\tag{3.14}$$

Theorem 2. *If Assumption 3 is met and additionally*

C1) $y(t)$ *and* $u_c(t)$ *are bounded,*

C2) $\Delta \notin L_2$,

then the identification laws (3.14) *ensure that:*

1) *the parametric errors* $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$, $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$, $\tilde{\psi}_b(t) = \hat{\psi}_b(t) - \psi_b(\theta)$ *converge asymptotically to zero:*

$$\lim_{t\to\infty} \tilde{\theta}(t) = 0, \;\; \lim_{t\to\infty} \tilde{\rho}(t) = 0, \;\; \lim_{t\to\infty} \tilde{\psi}_b(t) = 0,$$

2) $\tilde{\theta} \in L_2 \cap L_\infty$, $\tilde{\rho} \in L_2 \cap L_\infty$, $\tilde{\psi}_b \in L_2 \cap L_\infty$.

Proof of Theorem 2 is presented in Appendix.

As, according to the problem statement, the component $u_b(t)$ of the control law is stabilising and the polynomial $R_{cl}(\theta_{cl}, s)$ is Hurwitz one, then C1 requires to form a bounded compensation signal $u_c(t)$, for example, using a standard saturation function (see Section 3.3). Following proof from [32], a sufficient condition to meet C2 is that the reference signal $r(t)$ is sufficiently rich, *e.g.*, when $m = n - 1$, it has to include at least n different frequencies. Assumption 3 (inequality (2.6)) is met, for example, if the spectra of the reference $r(t)$ and the perturbation $f(t)$ signals do not have common frequencies. The convergence conditions for identification laws of type (3.14) are given in more detail in [32].

One of the freedom degrees of the used parameterization (2.3) and the whole proposed identification scheme is the choice of the type of regressor $\varphi(t)$. If the initial values of the unknown parameters estimates are such that $|u_c(t) + f(t)| > |f(t)|$ (in the sense of mean integral value or variance), then use of (2.3) results in a parameterization with a smaller perturbation value $w(t)$ in a similar sense. In contrast, if the initial values of the unknown parameters estimates are such that $|u_c(t) + f(t)| < |f(t)|$ is achieved even without parametric adaptation (in the sense of mean integral value or variance), then the choice of the following regressor

$$\varphi(t) = \begin{bmatrix} -\frac{\alpha_{n-1}^\top(s)}{\Lambda(s)} z(t) & \frac{\alpha_m^\top(s)}{\Lambda(s)} u_b(t) \end{bmatrix}^\top$$

allows one to obtain a parametrization with smaller value of the disturbance $w(t)$ in a similar sense.

The further parameterization as well as the premises and results of Theorem 2 do not depend on whether the regressor is calculated using $u_b(t)$ or $u(t)$.

The states of filters (3.9) lose their awareness to new values of $\vartheta(t)$ and $\psi(t)$ signals, and thus, the parameters θ too, at the rate of $\frac{\dot{F}(t)}{F(t)}$. It is currently not possible to completely prevent such loss of awareness, since the coefficient $\frac{\dot{F}(t)}{F(t)}$ ensures that the parametric error converges to zero in the presence of an external perturbation.

However, by redefining:

$$\begin{aligned} Y(t) &:= \frac{1}{T}\vartheta(t), \\ \Phi(t) &:= \frac{1}{T}\psi(t) \end{aligned} \tag{3.15}$$

an identification law can be derived, which properties are more acceptable for practical scenarios.

Theorem 3. *If Assumption 3 is met and additionally*

C1) $y(t)$ *and* $u_c(t)$ *are bounded,*

C2) *there exist scalars* $\Delta_{\rm UB} \geqslant \Delta_{\rm LB} > 0$ *such that* $\Delta_{\rm LB} \leqslant |\Delta(t)| \leqslant \Delta_{\rm UB}$, $\forall t \geqslant t_e$, *then there exist a scalar* $\delta_0 > 0$ *and a signal* $\delta_1 \in L_1$, $\lim\limits_{t\to\infty} \delta_1(t) = 0$ *such that*

$$\left\| \begin{bmatrix} \tilde{\theta}(t) \\ \tilde{\rho}(t) \\ \tilde{\psi}_b(t) \end{bmatrix} \right\| \leqslant \delta_1(t) + \delta_0 T^{-1},\ \forall t \geqslant t_0. \tag{3.16}$$

Proof of Theorem 3 can be found in Appendix.

Thus, the identification law (3.14) based on the signals (3.8) + (3.15) provides awareness to new values of the parameters θ, but at the same time parametric errors converge not to zero, but to its neighbourhood, which is proportional to the parameter $T > 0$, and under a more strict condition in comparison with $\Delta \notin L_2$.

3.3. Adaptive Auxiliary Loop

Being motivated by (3.1), (3.5a)–(3.5c), (3.6) and requirement C1 and using the estimates (3.14), the compensation component of the control law is formed as:

$$\begin{gathered} u_c(t) = \mathrm{sat}_{f_{\max}}\left\{-\hat{f}(t)\right\}, \\ \hat{f}(t) = \psi_a^\top(\Theta)\,\hat{h}_\varepsilon(t) + \left(\hat{\psi}_a^\top(t) - \psi_a^\top(\Theta)\right)\hat{h}_y(t), \end{gathered} \tag{3.17}$$

where

$$\begin{cases} \dot{\hat{H}}_i^\varepsilon(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)\hat{H}_i^\varepsilon(t) + e_{n+1}\left(K_0(t)\hat{\varepsilon}_f(t) - \hat{v}_\varepsilon(t)\right) \\ \hat{h}_i^\varepsilon(t) = e_{i+1}^\top \hat{H}_i^\varepsilon(t),\ \hat{v}_\varepsilon(t) = \sum\limits_{j=0}^{i} \hat{v}_j^\varepsilon(t,\ M_0,\ K_0), \end{cases}$$
$$\forall i = 0, \ldots, n, \tag{3.18a}$$
$$\begin{cases} \dot{\hat{H}}_i^y(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)\hat{H}_i^y(t) + e_{n+1}\left(K_0(t)\hat{y}_f(t) - \hat{v}_y(t)\right) \\ \hat{h}_i^y(t) = e_{i+1}^\top \hat{H}_i^y(t),\ \hat{v}_y(t) = \sum\limits_{j=0}^{i} \hat{v}_j^y(t,\ M_0,\ K_0), \end{cases}$$

$$\begin{aligned} \dot{\hat{\xi}}_\varepsilon(t) &= \left(A_0 - \hat{\psi}_b(t)\,e_1^\top\right)\hat{\xi}_\varepsilon(t) + \hat{\rho}(t)\,e_m\varepsilon(t), \\ &\begin{cases} \hat{\varepsilon}_f(t) = e_1^\top \hat{\xi}_\varepsilon(t),\ \text{if } m \geqslant 1 \\ \hat{\varepsilon}_f(t) = \hat{\rho}(t)\,\varepsilon(t),\ \text{if } m = 0, \end{cases} \\ \dot{\hat{\xi}}_y(t) &= \left(A_0 - \hat{\psi}_b(t)\,e_1^\top\right)\hat{\xi}_y(t) + \hat{\rho}(t)\,e_m y(t), \\ &\begin{cases} \hat{y}_f(t) = e_1^\top \hat{\xi}_y(t),\ \text{if } m \geqslant 1 \\ \hat{y}_f(t) = \hat{\rho}(t)\,y(t),\ \text{if } m = 0, \end{cases} \end{aligned} \tag{3.18b}$$

$$\begin{cases} \dot{\hat{x}}(t) = A_0\hat{x}(t) - \Theta\hat{y}(t) + \hat{\theta}_b(t)\,u(t) \\ \hat{y}(t) = e_1^\top \hat{x}(t), \end{cases} \tag{3.18c}$$

and $\varepsilon(t) = y(t) - \hat{y}(t)$, $\hat{\psi}_a(t) := \psi_a\left(\hat{\theta}_a\right)$, $\hat{\theta}_a(t) = \mathcal{L}_a\hat{\theta}(t)$, $\mathrm{sat}_{f_{\max}}\{.\}$ is a conventional saturation function to bound the absolute value of the signal $u_c(t)$ by $f_{\max}$. The signals $\hat{v}_\varepsilon(t)$ and $\hat{v}_y(t)$ are calculated via equations given in Proposition 1.

Theorem 4. *If*

1) *Assumptions 1–3 are met and the premise* C2 *from Theorem 2 is satisfied,*
2) $|f(t)| < f_{\max}$ *for all* $t \geqslant t_0$ *and a sufficiently large scalar* $f_{\max} > 0$,

then the control signal (2.2) + (3.17) + (3.14) *ensures* $\lim\limits_{t\to\infty} |\tilde{y}(t)| = 0$.

Proof of Theorem 4 is given in Appendix.

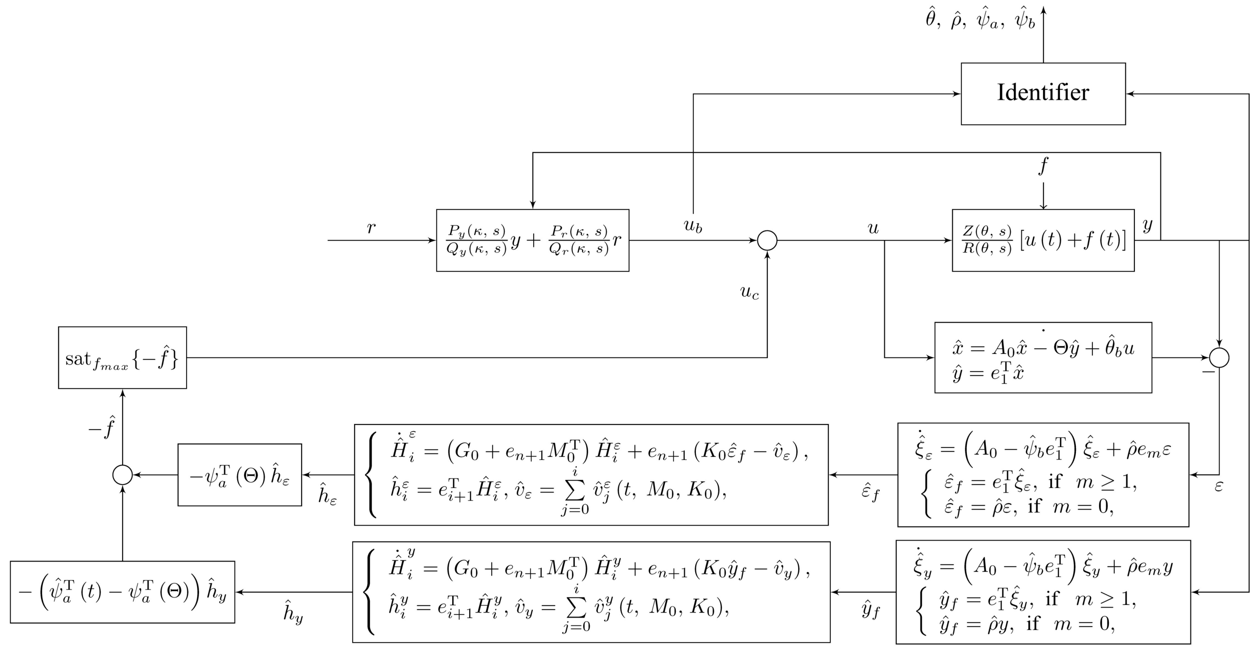

Fig. 1. Structural scheme of adaptive auxiliary loop.

The application of time-varying filters (3.18a) may be inappropriate for practical scenarios, for example, due to the requirement of the control signal noise resistance or inaccuracies related to the discretization/numerical solution of time-varying differential equations (3.18a). Therefore, in Theorem 5 we investigate the stability of the closed-loop control system for the case when $\mu\left(t\right)=\text{const}$ and Assumption 2 is not satisfied (in the sense that $\lambda\notin L_p$). Note that the situation when $\mu\left(t\right)=const$ but Assumption 2 is met (*i.e.*, $\lambda\notin L_p$) has already been considered in Theorem 4.

Theorem 5. *If*

1) *Assumptions 1, 3 are met and the premise* C2 *from Theorem 2 is satisfied,*
2) $\mu\left(t\right)=\mu>0,\ \lambda\in L_\infty,$
3) $\left|f\left(t\right)\right|<f_{\max}$ *for all* $t\geqslant t_0$ *and sufficiently large scalar* $f_{\max}>0,$

then the control signal (2.2) + (3.17) + (3.14) *ensures* $\lim\limits_{t\to\infty}\left|\tilde{y}\left(t\right)\right|\leqslant\epsilon$ *for arbitrarily small value of* $\epsilon>0.$

Proof of Theorem is postponed to Appendix.

Therefore, if $(n+1)$ derivatives of the perturbation are bounded, then the proposed adaptive compensation system (3.17), (3.18a)–(3.18c), (3.9)+(3.14) of the external disturbance with time-invariant observers of the derivatives (3.18a) ensures convergence of the tracking error to an arbitrarily small neighbourhood of zero.

Now we use the identifier (3.14) based on (3.15) instead of (3.9) to adjust the parameters of the compensator (3.17), (3.18a)–(3.18c).

Theorem 6. *If*

1) *Assumptions 1, 3 are met and the premise* C2 *from Theorem 3 is satisfied,*
2) $\mu\left(t\right)=\mu>0,\ \lambda\in L_\infty,$
3) $\left|f\left(t\right)\right|<f_{\max}$ *for all* $t\geqslant t_0$ *and sufficiently large scalar* $f_{\max}>0,$

then the control signal $(2.2)+(3.17)$ *with the identification laws* $(3.14)+(3.15)$ *ensures* $\lim\limits_{t\to\infty}|\tilde{y}(t)|\leqslant\epsilon$ *for an arbitrarily small scalar* $\epsilon>0.$

Proof of Theorem 6 can be found in Appendix.

The proposed adaptive system to compensate for an external bounded perturbation consists of an adaptive auxiliary model (3.18c), adaptive filtering algorithms (3.18b), filtered derivative observers with high-gain feedback (3.18a), compensation law (3.17) and identification algorithms (3.14) based on the signals calculated with the help of the instrumental model (2.5) and filters (2.3), (3.8), (3.9) or (2.3), (3.8), (3.15). The structural diagram of such a control system is shown in Fig. 1.

Unlike the existing solutions [2–26], if the conditions of parametric convergence are met, the proposed adaptive compensator separates estimation of the additive and parametric perturbations and asymptotically compensates for the perturbation $f(t)$, which affects the system with unknown parameters. At the same time, the stabilising component of the controller is designed independently of the compensation one, which allows one to obtain a control system with two degrees of freedom. In fact, three different schemes of external perturbation compensation are proposed in this study, which can be classified as follows:

—time-varying observers of filtered derivatives (3.18a) + the identifier (3.14) based on the signals obtained with the help of the filtering with averaging (3.9),

—time-invariant observers of the filtered derivatives (3.18a) + the identifier (3.14) based on the signals obtained with the help of the filtering with averaging (3.9),

—time-invariant observers of the filtered derivatives (3.18a) + the identifier (3.14) based on the signals obtained with the help of the sliding window filtering (3.8) + (3.15).

The first two schemes are of theoretical significance, but due to the loss of awareness of the (3.9) filters to the unknown parameters changes and the potentially infinitely large gain of the filtered derivatives observers (3.18a), they are of little practical value. The third scheme is free from the disadvantages of the first two schemes, but ensures convergence of the tracking error only to a bounded neighbourhood of the equilibrium point and under a stricter condition (C2 from Theorem 3 instead of C2 from Theorem 2).

Remark 5. The use of time-varying derivative observers (3.18a) together with an identifier based on signals obtained by sliding window filtering (3.15) is not reasonable, as, due to the properties of the identifier, the convergence of the tracking error will be ensured only to a bounded set.

4. NUMERICAL EXPERIMENTS

A differential equation has been considered:

$$m\frac{d^2}{dt^2}y(t) = \mathcal{F}(y,\ u,\ t).$$

We assumed that $\frac{1}{m}\mathcal{F}(y,\ u,\ t)$ could be approximated as:

$$\frac{1}{m}\mathcal{F}(y,\ u,\ t) := \theta_2\frac{d}{dt}y(t) + \theta_1 y(t) + \theta_3(u + f(t)),\ \theta_3 := \frac{1}{m},$$

from which, owing to $s[.] := \frac{d}{dt}[.]$, the second-order dynamic system was obtained:

$$y(t) = \frac{\theta_3}{s^2 + \theta_2 s + \theta_1}[u(t) + f(t)]. \tag{4.1}$$

The stabilizing component of the control law was defined as a PID-controller with proper differential summand:

$$u_b(t) = K_P(r(t) - y(t)) + \frac{K_I}{s}[r(t) - y(t)] + \frac{K_D s}{K_F s + 1}[r(t) - y(t)]. \tag{4.2}$$

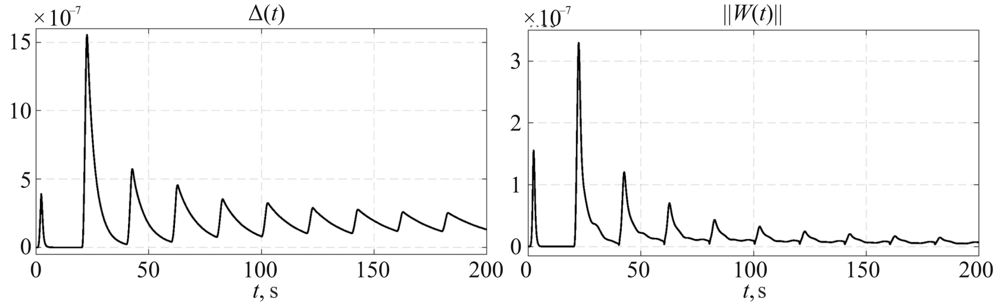

Fig. 2. Behavior of $\Delta(t)$ and $\|\mathcal{W}(t)\|$.

The parameters of the system (4.1) and the control law (4.2) were chosen as:

$$\begin{gathered}\theta_1 = 2,\quad \theta_2 = 5,\quad \theta_3 = 1,\\ K_P = 6,\quad K_I = 2.5,\quad K_D = 1.5,\quad K_F = 0.01,\\ f(t) = 2.5 + 1.25\cos(\pi t) + 2.5\sin(0.3\pi t),\\ r(t) = 10\mathrm{sgn}(\sin(0.05\pi t)).\end{gathered} \tag{4.3}$$

The parameters of the adaptive auxiliary loop (3.18a)–(3.18c), the parametrization (2.3), the instrumental model (2.5), the filters (3.8), (3.9) and the identification laws (3.14) were set as follows:

$$\begin{gathered}\Theta = \begin{bmatrix}20 & 100\end{bmatrix}^\top,\ m_0 = 1,\ m_1 = 3,\ m_3 = 3,\ \mu = 10^4,\\ \Lambda(s) = s^2 + 20s + 100,\ Z(\theta_{iv},\ s) = 4,\\ R(\theta_{iv},\ s) = s^2 + 2s + 4,\ p = 2,\ T = 4,\\ \hat{\theta}_0 = \begin{bmatrix}2 & 4 & 2\end{bmatrix}^\top,\ \hat{\rho}_0 = 0.5,\ \gamma = \gamma_\rho = 10^{13},\ \overline{f}_{\max} = 10.\end{gathered} \tag{4.4}$$

The parameters of the adaptive auxiliary loop (3.18a)–(3.18c), the parameterization (2.3) and the instrumental model (2.5) were chosen to guarantee the stability of the corresponding differential equations. The values of gains $\gamma,\ \gamma_\rho$ were picked by trial and error so as to ensure approximately the following proportionality:

$$\gamma \sim \frac{1}{\Delta^2(t)},\ \gamma_\rho \sim \frac{1}{\mathcal{M}_\rho^2(t)}. \tag{4.5}$$

Considering practical scenarios, we need *a priori* data on the amplitude values of the regressor $\Delta(t)$ calculated for typical system trajectories at fixed p and T to choose $\gamma,\ \gamma_\rho,\ \gamma_\rho,\ \gamma_{\psi_b}$. In case such information is not available, the parameters (4.5) have to be picked online by trial and error.

Figure 2 presents the behavior of the regressor $\Delta(t)$ and the norm of the disturbance $\mathcal{W}(t)$ from the regression equation (3.12).

The transients presented in Fig. 2 demonstrate that throughout the simulation, starting from $t = T = 4$, the regressor $\Delta(t)$ was bounded away from zero and the perturbation $\mathcal{W}(t)$ was asymptotically decreasing. According to the analysis from [32], these two observations verify that Assumption 3 and the premise C2 from Theorem 2 were met.

Figure 3a depicts behavior of the tracking error $|\tilde{y}(t)|$. In Fig. 3b transients of the compensation error $|u_c(t) + f(t)|$ are given. Figures 3c and 3d are to show behavior of estimates $\hat{\theta}(t)$ and $\hat{\rho}(t)$, respectively.

The presented transients illustrate the result proved in Theorem 4. If the parametric convergence conditions C2 from Theorem 2 and Assumption 3 are met (see comments to Fig. 2), then the asymptotic convergence of the estimates $\hat{\theta}(t)$ and $\hat{\rho}(t)$ to the unknown parameters θ and ρ is

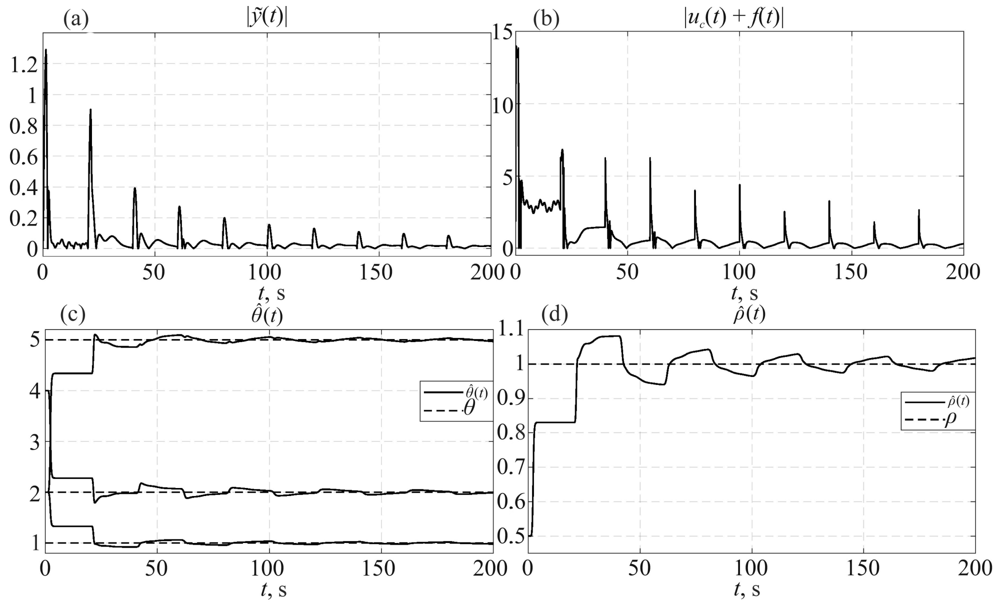

Fig. 3. Behavior of $|\tilde{y}(t)|$, $|u_c(t) + f(t)|$ and $\hat{\theta}(t)$, $\hat{\rho}(t)$.

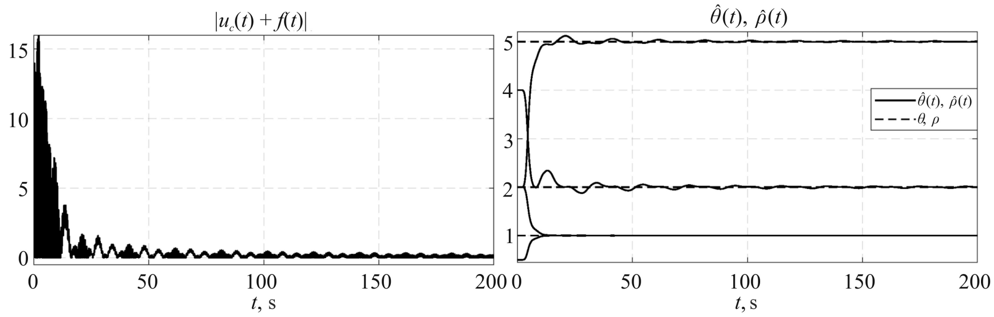

Fig. 4. Behavior of $|u_c(t) + f(t)|$ and $\hat{\theta}(t)$, $\hat{\rho}(t)$.

ensured, which leads to the asymptotic convergence to zero of the compensation error $u_c(t) + f(t)$, which, in its turn, results in asymptotic compensation of the effect of the perturbation $f(t)$ on the transient quality. Note that from the practical point of view, due to the use of the integral summand in the control signal (4.2), it is sufficient to guarantee $u_c(t) + f(t) \approx \text{const}$ to compensate for the perturbation $f(t)$. Considering the conducted experiment, the parametric error convergence was achieved when we used a typical for practical scenarios reference signal in the form of a rectangular periodic signal. This result allows one to conclude that the conditions of parametric error convergence are not restrictive and met for typical practical cases.

The next aim was to demonstrate that, if the reference signal had a special form, then the transient quality for $\hat{\theta}(t)$, $\hat{\rho}(t)$ and $u_c(t) + f(t)$ could be improved. So $r(t)$ and gains γ, γ_ρ were chosen as:

$$r(t) = 10\left[\sin(0.2\pi t) + \sin(3\pi t)\right], \; \gamma = \gamma_\rho = 10^{10}, \tag{4.6}$$

whereas all other parameters values were set in accordance with (4.3) and (4.4).

The reference signal (4.6) spectrum has no common frequencies with the one of the disturbance, and $r(t)$ includes $n = 2$ frequencies. According to [32], together with the boundedness of $y(t)$, $u_c(t)$, it immediately results in satisfaction of the premise C2 from Theorem 2 and Assumption 3.

Figure 4 depicts behavior of $u_c(t) + f(t)$ and $\hat{\theta}(t)$, $\hat{\rho}(t)$ for the experiment with the reference signal (4.6).

Such choice of the reference signal improved the transient quality for $\hat{\theta}(t)$, $\hat{\rho}(t)$ and $u_c(t)+f(t)$. Therefore, the conducted numerical experiments validated the theoretical conclusions made in this study and illustrated the properties of the proposed system for different reference signals.

Remark 6. Results of numerical experiments, which were not included in the paper, showed that for any choice of initial conditions $\hat{\theta}_0, \ \hat{\rho}_0$ the proposed solution provided asymptotic perturbation compensation. However, the transients quality could be arbitrarily poor in this case. Therefore, for practical scenarios, it is recommended to make the initial conditions $\hat{\theta}_0, \ \hat{\rho}_0$ equal to the 'nominal' parameters of the system, for example, the ones for which the stabilising component of the control law was designed. In this case, it is possible not only to ensure that the goal of (2.8) is achieved, but also to obtain an admissible transient quality.

5. CONCLUSION

A method of output-based compensation of a bounded additive perturbation affecting a linear minimum-phase system with unknown parameters is developed. The basis of the proposed solution is the perturbation compensation algorithm grounded on A.M. Tsykunov's auxiliary loop method [26, p. 196], which requires knowledge of the system parameters for asymptotic estimation and compensation of the additive perturbation. To relax this requirement, the identification law based on the method of instrumental variables and the DREM procedure is used to provide exact online asymptotic identification of the system unknown parameters in a closed loop. The convergence conditions of the parameter identification process are:

C1) boundedness of the compensation signal (for example, via $\text{sat}\{.\}$ function),
C2) sufficiently rich reference signal,
C3) no common frequencies in spectra of the reference and disturbance signals.

The obtained estimates are used instead of the unknown parameters in the perturbation compensation algorithm based on the auxiliary loop method [26]. It is shown that the substitution of unknown parameters with their estimates is feasible in the sense that under the conditions C1)–C3) the proposed system of perturbation adaptive compensation, as well as its non-adaptive equivalent, provides asymptotic estimation and compensation of the disturbance.

The properties of the proposed solution are demonstrated via numerical simulation. It is shown that the conditions C1)–C3) are not restrictive and met for typical control systems with PI-, PID-controllers and conventional reference signals. The authors believe that the proposed approach has a potential for practical application, but note that the method has a drawback related to the need for trial and error selection of some parameters of the algorithm for identification and estimation of unmeasured derivatives.

The scope of further research is to improve the quality of transients for perturbation estimation and increase the convergence rate of the unknown parameters estimates to their true values.

APPENDIX A

This appendix contains auxiliary lemmas, which are axiomatically used to prove main result of this study.

Lemma A1. *For any stable and proper operator* $\mathcal{H}(t, \ s)[.]$ *and corresponding signal* $y(t) = \mathcal{H}(t, \ s)[u(t)]$ *the following holds:* $u \in L_p \Rightarrow y \in L_p, \ p \in [1, \ \infty]$.

Proof is presented in [35, p. 75].

Lemma A2. *If* $\dot{f} \in L_\infty$ *and* $f \in L_p, \ p \in (0, \ \infty)$*, then* $\lim\limits_{t \to \infty} f(t) = 0$.

Proof is given in [35, p. 80].

Lemma A3. *Consider the scalar system defined by*

$$\dot{x}(t) = -a^2(t)\,x(t) + b(t),\ x(t_0) = x_0,$$

where $x(t) \in \mathbb{R}$ *and* $a,\ b\colon \mathbb{R}_+ \mapsto \mathbb{R}$ *are piecewise continuous bounded functions. If* $a \notin L_2$ *and* $b \in L_1$, *then* $\lim\limits_{t\to\infty} x(t) = 0$.

Proof can be found in [36, Section 3.A.1].

Lemma A4. *Consider the nonnegative scalar functions* $f\colon [t_0,\ \infty) \mapsto \mathbb{R}$, $g\colon [t_0,\ \infty) \mapsto \mathbb{R}$. *If* $f(t) \leqslant g(t)$ *for all* $t \geqslant t_0$ *and* $g \in L_p$, $p \in (0,\ \infty)$, *then* $f \in L_p$.

Proof is presented in [28, p. 74].

Lemma A5. *If* $f \in L_p$, $1 \leqslant p < \infty$, *then* $g(t) = H(s)[f(t)] \in L_\infty$ *and* $\lim\limits_{t\to\infty} g(t) = 0$ *for any stable and strictly proper transfer function* $H(s)[.]$.

Proof is given in [35, p. 83].

APPENDIX B

Proof of Statement 1. A linear time-varying operator is defined as:

$$\mathcal{H}(t,s)[.] = e_1^\top \left(sI_{n+1} - G_0 - e_{n+1}M_0^\top(t)\right)^{-1} e_{n+1}K_0(t)[.].$$

In order to prove the proposition under consideration, firstly, an auxiliary lemma is proved for the above-given time-varying operator.

Lemma B1. *Define signals* $\left(s[.] := \frac{d}{dt}[.]\right)$:

$$Y^*(t) = \mathcal{H}(t,s)\left[s^i[U(t)]\right], \tag{B.1a}$$

$$Y_i(t) = e_{i+1}^\top \left(sI_{n+1} - G_0 - e_{n+1}M_0^\top(t)\right)^{-1} e_{n+1}K_0(t)[U(t) - v(t)], \tag{B.1b}$$

where

$$\begin{aligned} &v(t) = \sum_{j=0}^{i} v_j(t,\ M_0,\ K_0),\\ &v_0(t,\ M_0,\ K_0) = 0,\\ &v_1(t,\ M_0,\ K_0) = s^{-1}\left[s\left[M_0^\top(t)\right]X(t)\right] + s^{-1}\left[s\left[K_0(t)\right]U(t)\right],\\ &\vdots\\ &v_j(t,\ M_0,\ K_0) = v_{j-1}(t,\ M_0,\ K_0) - s^{-1}\left[v_{j-1}\left(t,\ \dot{M}_0,\ \dot{K}_0\right)\right],\ j = 2,\dots i, \end{aligned} \tag{B.2}$$

and $X(t) = \left(sI_{n+1} - G_0 - e_{n+1}M_0^\top(t)\right)^{-1} e_{n+1}K_0(t)[U(t) - v(t)]$.

Then for all $i = 0,\dots,n$ *and* $t \geqslant t_0$ *it holds that* $Y^*(t) = Y_i(t)$.

Proof. Equations (B.1a) and (B.1b) are represented in the state-space form:

$$\begin{cases} \dot{X}^*(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)X^*(t) + e_{n+1}K_0(t)\,s^i[U(t)]\\ Y^*(t) = e_1^\top X^*(t), \end{cases}$$

$$\begin{cases} \dot{X}(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)X(t) + e_{n+1}\left(K_0(t)U(t) - v(t)\right)\\ Y_i(t) = e_{i+1}^\top X(t). \end{cases}$$

Owing to the structure of the matrix $G_0 + e_{n+1}M_0^\top(t)$, it is easy to check that the following equations hold:

$$\begin{aligned}
&e_1^\top X(t) = Y_0(t),\\
&e_1^\top s[X(t)] = e_1^\top \left(G_0 + e_{n+1}M_0^\top(t)\right)X(t) = e_2^\top X(t) = Y_1(t),\\
&e_1^\top s^2[X(t)] = e_1^\top s\left[\left(G_0 + e_{n+1}M_0^\top(t)\right)X(t)\right] = e_1^\top\left(G_0 + e_{n+1}M_0^\top(t)\right)\dot{X}(t)\\
&\qquad\qquad = e_1^\top\left(G_0 + e_{n+1}M_0^\top(t)\right)^2 X(t) = e_2^\top s[X(t)] = e_3^\top X(t) = Y_2(t),\\
&\vdots\\
&e_1^\top s^i[X(t)] = e_i^\top s[X(t)] = e_{i+1}^\top X(t) = Y_i(t),\quad i = 0,\ldots,n,
\end{aligned}$$

which allows one to define the error vector as:

$$\begin{aligned}
E(t) &= s^i[X(t)] - X^*(t),\\
e_1^\top E(t) &= Y_i(t) - Y^*(t).
\end{aligned}$$

Equations for $s^{i+1}[X(t)]$ for all $i = 0,\ldots,n$ are written as:

$$\begin{aligned}
s[X(t)] &= \left(G_0 + e_{n+1}M_0^\top(t)\right)X(t) + e_{n+1}\left(\tilde{v}_0(t) + K_0(t)U(t) - \sum_{j=1}^{i} v_j(t)\right),\\
s^2[X(t)] &= \left(G_0 + e_{n+1}M_0^\top(t)\right)s[X(t)] + e_{n+1}\left(\tilde{v}_1(t) + K_0(t)s[U(t)] - s\left[\sum_{j=2}^{i} v_j(t)\right]\right),\\
s^3[X(t)] &= \left(G_0 + e_{n+1}M_0^\top(t)\right)s^2[X(t)] + e_{n+1}\left(\tilde{v}_2(t) + K_0(t)s^2[U(t)] - s^2\left[\sum_{j=3}^{i} v_j(t)\right]\right),\\
s^4[X(t)] &= \left(G_0 + e_{n+1}M_0^\top(t)\right)s^3[X(t)] + e_{n+1}\left(\tilde{v}_3(t) + K_0(t)s^3[U(t)] - s^3\left[\sum_{j=4}^{i} v_j(t)\right]\right),\\
&\vdots\\
s^{i+1}[X(t)] &= \left(G_0 + e_{n+1}M_0^\top(t)\right)s^i[X(t)] + e_{n+1}\left(\tilde{v}_i(t) + K_0(t)s^i[U(t)]\right),
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{v}_0(t) = v_0(t),\\
&\tilde{v}_1(t) = \dot{\tilde{v}}_0(t) + s\left[M_0^\top(t)\right]X(t) + s[K_0(t)]U(t) - s[v_1(t)],\\
&\tilde{v}_2(t) = \dot{\tilde{v}}_1(t) + s\left[M_0^\top(t)\right]s[X(t)] + s[K_0(t)]s[U(t)] - s^2[v_2(t)],\\
&\tilde{v}_3(t) = \dot{\tilde{v}}_2(t) + s\left[M_0^\top(t)\right]s^2[X(t)] + s[K_0(t)]s^2[U(t)] - s^3[v_3(t)],\\
&\vdots\\
&\tilde{v}_i(t) = \dot{\tilde{v}}_{i-1}(t) + s\left[M_0^\top(t)\right]s^{i-1}[X(t)] + s[K_0(t)]s^{i-1}[U(t)] - s^i[v_i(t)],\ i = 1,\ldots,\ n.
\end{aligned}$$

Owing to the recurrent sequence (B.2), it is obtained:

$$\begin{aligned}
&\tilde{v}_0(t) = v_0(t),\\
&s[v_1(t)] = s\left[M_0^\top(t)\right]X(t) + s[K_0(t)]U(t),\\
&s^2[v_2(t)] = s^2\left[M_0^\top(t)\right]X(t) + s^2[K_0(t)]U(t) + s\left[M_0^\top(t)\right]s[X(t)]\\
&\qquad + s[K_0(t)]s[U(t)] - s^2\left[M_0^\top(t)\right]X(t) - s^2[K_0(t)]U(t)\\
&\qquad = s\left[M_0^\top(t)\right]s[X(t)] + s[K_0(t)]s[U(t)],\\
&\vdots\\
&s^j[v_j(t)] = s\left[M_0^\top(t)\right]s^{j-1}[X(t)] + s[K_0(t)]s^{j-1}[U(t)], \quad j = 3, \dots i,
\end{aligned}$$

from which $\tilde{v}_i(t) = 0$ for all $i = 0, \dots, n$.

Then the derivative of $E(t)$ is written as:

$$\begin{aligned}
\dot{E}(t) &= s^{i+1}[X(t)] - \dot{X}^*(t)\\
&= \left(G_0 + e_{n+1}M_0^\top(t)\right)s^i[X(t)] + e_{n+1}\left(\tilde{v}_i(t) + K_0(t)s^i[U(t)]\right)\\
&\quad - \left(G_0 + e_{n+1}M_0^\top(t)\right)X^*(t) - e_{n+1}K_0(t)s^i[U(t)]\\
&= \left(G_0 + e_{n+1}M_0^\top(t)\right)E(t),\ E(t_0) = 0_{n+1},
\end{aligned}$$

and therefore, owing to the facts that $G_0 + e_{n+1}M_0^\top(t)$ is a Hurwitz matrix and $E(t_0) = 0_{n+1}$, it is obtained that $Y^*(t) = Y_i(t)$ for all $i = 0, \dots, n$ and $t \geqslant t_0$.

Proof of Lemma B1 is completed.

Now we are in position to continue proof of Proposition 1. Having applied the operator $\mathcal{H}(t,s)[.]$ to the left- and right-hand sides of (3.5a), it is obtained:

$$f_f(t) := \mathcal{H}(t,s)[f(t)] = \psi_a^\top(\Theta)h_\varepsilon(t) + \left(\psi_a^\top(\theta_a) - \psi_a^\top(\Theta)\right)h_y(t),$$

where

$$h_y(t) = \begin{bmatrix}\mathcal{H}(t,s)[y_f(t)]\\ \vdots\\ \mathcal{H}(t,s)\left[s^{n-1}[y_f(t)]\right]\\ \mathcal{H}(t,s)[s^n[y_f(t)]]\end{bmatrix},\ h_\varepsilon(t) = \begin{bmatrix}\mathcal{H}(t,s)[\varepsilon_f(t)]\\ \vdots\\ \mathcal{H}(t,s)\left[s^{n-1}[\varepsilon_f(t)]\right]\\ \mathcal{H}(t,s)[s^n[\varepsilon_f(t)]]\end{bmatrix}.$$

The application of Lemma B1 yields:

$$h_y(t) = \begin{bmatrix}\mathcal{H}(t,s)[y_f(t)]\\ \vdots\\ \mathcal{H}(t,s)\left[s^{n-1}[y_f(t)]\right]\\ \mathcal{H}(t,s)[s^n[y_f(t)]]\end{bmatrix} = \begin{bmatrix}h_0^y(t)\\ \vdots\\ h_i^y(t)\\ \vdots\\ h_n^y(t)\end{bmatrix},\ h_\varepsilon(t) = \begin{bmatrix}\mathcal{H}(t,s)[\varepsilon_f(t)]\\ \vdots\\ \mathcal{H}(t,s)\left[s^{n-1}[\varepsilon_f(t)]\right]\\ \mathcal{H}(t,s)[s^n[\varepsilon_f(t)]]\end{bmatrix} = \begin{bmatrix}h_0^\varepsilon(t)\\ \vdots\\ h_i^\varepsilon(t)\\ \vdots\\ h_n^\varepsilon(t)\end{bmatrix},$$

so, consequently, the definition (3.6) is obtained.

The original $f(t)$ and filtered $f_f(t)$ disturbances are represented in the state-space form:

$$\begin{cases} \dot{\mathcal{F}}(t) = G_0\mathcal{F}(t) + e_{n+1}s^{n+1}[f(t)] \\ f(t) = e_1^\top \mathcal{F}(t), \end{cases}$$

$$\begin{cases} \dot{\mathcal{F}}_f(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)\mathcal{F}_f(t) + e_{n+1}K_0(t)[f(t)] \\ f_f(t) = e_1^\top \mathcal{F}_f(t). \end{cases}$$

Then the error $\tilde{f}(t)$ satisfies the below-given equation:

$$\begin{gathered} \dot{\tilde{\mathcal{F}}} = \left(G_0 + e_{n+1}M_0^\top(t)\right)\mathcal{F}_f(t) + e_{n+1}K_0(t)[f(t)] \\ - \left(G_0 \pm e_{n+1}M_0^\top(t)\right)\mathcal{F}(t) - e_{n+1}s^{n+1}[f(t)] = \left(G_0 + e_{n+1}M_0^\top(t)\right)\tilde{\mathcal{F}}(t) \\ + e_{n+1}K_0(t)\left(f(t) - \frac{1}{m_0\mu^{n+1}(t)}\left(s^{n+1}[f] + \sum_{i=0}^{n} m_i\mu^{n-(i-1)}(t)s^i[f(t)]\right)\right) \\ = \left(G_0 + e_{n+1}M_0^\top(t)\right)\tilde{\mathcal{F}}(t) + e_{n+1}K_0(t)\lambda(t), \\ \tilde{f}(t) = e_1^\top\tilde{\mathcal{F}}(t), \end{gathered}$$

or, representing it in the operator form, we have:

$$\tilde{f}(t) = \mathcal{H}(t,s)[\lambda(t)]. \tag{B.3}$$

In accordance with Assumption 2, it holds that $\lambda \in L_p \cap L_\infty$ for some $p \in [1, \infty)$, and then application of Lemma A1 allows one to obtain $\tilde{f} \in L_p \cap L_\infty$.

Equation (B.3) is rewritten in the observer canonical form:

$$\begin{cases} \dot{E}(t) = \left(G_0 + \mathcal{I}_{n+1}M_0(t)e_1^\top\right)E(t) + e_{n+1}K_0(t)\lambda(t) \\ \tilde{f}(t) = e_1^\top E(t). \end{cases} \tag{B.4}$$

The normalized error is defined as

$$\eta(t) = \Gamma^{-1}(t)E(t),$$

where $\Gamma(t) = \text{diag}\{1, \mu(t), \ldots, \mu^n(t)\}$, and therefore, $\tilde{f}(t) = e_1^\top\eta(t) = e_1^\top E(t)$.

Having differentiated $\eta(t)$ and used (B.4), it is obtained:

$$\begin{aligned} \dot{\eta}(t) &= \frac{d\Gamma^{-1}(t)}{dt}E(t) + \Gamma^{-1}(t)\left(G_0 + \mathcal{I}_{n+1}M_0(t)e_1^\top\right)E(t) + \Gamma^{-1}(t)e_{n+1}K_0(t)\lambda(t) \\ &= \left(\frac{d\Gamma^{-1}(t)}{dt}\Gamma(t) + \mu(t)G\right)\eta(t) + e_{n+1}m_0\mu(t)\lambda(t),\ G = G_0 - \mathcal{I}_{n+1}Me_1^\top, \end{aligned}$$

where the following equalities are used (they can be easily checked via substitution):

$$\Gamma^{-1}(t)G_0\Gamma(t) = \mu(t)G_0,\quad \Gamma^{-1}(t)\mathcal{I}_{n+1}M_0(t) = -\mu(t)\mathcal{I}_{n+1}M,\quad e_1^\top\Gamma(t) = e_1^\top.$$

According to Assumption 2, it holds that $\mu\lambda \in L_\infty$, and, as the autonomous differential equation

$$\dot{x}(t) = \left(\frac{d\Gamma^{-1}(t)}{dt}\Gamma(t) + \mu(t)G\right)x(t),\ x(t_0) = x_0$$

is asymptotically stable, then $\mu\lambda \in L_\infty \Rightarrow \dot{\eta} \in L_\infty$, therefore, owing to $\dot{\tilde{f}}(t) = \dot{\eta}_1(t)$ it holds that $\dot{\tilde{f}} \in L_\infty$, and, following Lemma A2, we have $\lim\limits_{t\to\infty} \tilde{f}(t) = 0$.

Proof of Theorem 1. Having substituted (2.2) + (3.7) into (2.1) and subtracted (2.7) from the obtained result, it is written:

$$\tilde{y}(t) = W_{cl}(\theta_{cl},\ s)\left[\tilde{f}(t)\right].$$

According to Proposition 1, we have $\tilde{f} \in L_p \cap L_\infty$ for $p \in [1,\ \infty)$, then $\tilde{y} \in L_\infty$ and, following Lemma A5, it is obtained that $\lim\limits_{t\to\infty} \tilde{y}(t) = 0$.

Proof of Theorem 2. The identification laws (3.14) are rewritten in general terms:

$$\dot{\hat{\kappa}}(t) = -\gamma\mathcal{M}_\kappa(t)\left(\mathcal{M}_\kappa(t)\hat{\kappa}(t) - \mathcal{Y}_\kappa(t)\right),\ \hat{\kappa}(t_0) = \hat{\kappa}_0, \tag{B.5}$$

where $\kappa \in \{\theta,\ \rho\}$, $\hat{\kappa} \in \left\{\hat{\theta},\ \hat{\rho}\right\}$ and $\mathcal{Y}_\kappa(t) = \mathcal{M}_\kappa(t)\kappa + \mathcal{W}_\kappa(t)$.

Taking into consideration Proposition 5 from [32], if C1 and Assumption 3 are met, then $\mathcal{W} \in L_2$. Thus, as $\mathcal{M}_\rho \in L_\infty$, $\mathcal{Y}_\rho \in L_\infty$ (owing to C1), then on the basis of (3.12) and (3.13) we have $\mathcal{W}_\rho \in L_2$. As, owing to (3.13), it holds that

$$\Delta \notin L_2 \Rightarrow \mathcal{M}_\rho \notin L_2,$$

then Theorem 2 can be proved via analysis of properties of the general law (B.5) in case the conditions $\mathcal{W}_\kappa \in L_2$ and $\mathcal{M}_\kappa \notin L_2$ are satisfied.

Considering Theorem 2 from [32] and Proposition 1 from [37], if $\mathcal{W}_\kappa \in L_2$ and $\mathcal{M}_\kappa \notin L_2$, then $\lim\limits_{t\to\infty} \tilde{\kappa}(t) = 0$, $\tilde{\kappa} \in L_\infty$. Therefore, to complete proof, we need to show $\tilde{\kappa} \in L_2$.

To that end, the following upper bound of the derivative of the function $V = \frac{1}{2}\tilde{\kappa}^\top\tilde{\kappa}$ is obtained:

$$\begin{aligned}\dot{V}(t) &\leqslant -\gamma\tilde{\kappa}^\top\mathcal{M}_\kappa^2(t)\tilde{\kappa} + \gamma\tilde{\kappa}^\top\mathcal{M}_\kappa(t)\mathcal{W}_\kappa(t)\\ &\leqslant -\gamma(1-\chi)\tilde{\kappa}^\top\mathcal{M}_\kappa^2(t)\tilde{\kappa} + \gamma\chi^{-1}\|\mathcal{W}_\kappa(t)\|^2\\ &\leqslant -2\gamma\mathcal{M}_\kappa^2(t)(1-\chi)V(t) + \gamma\chi^{-1}\|\mathcal{W}_\kappa(t)\|^2,\end{aligned} \tag{B.6}$$

where $\chi \in (0, 1)$.

Having integrated (B.6), it is written:

$$V(t) \leqslant V(t_0) - \gamma(1-\chi)\int\limits_{t_0}^{t}\mathcal{M}_\kappa^2(s)\|\tilde{\kappa}(s)\|^2 ds + \gamma\chi^{-1}\int\limits_{t_0}^{t}\|\mathcal{W}_\kappa(s)\|^2 ds.$$

As $V \in L_\infty$ (it was shown above) and $\mathcal{W}_\kappa \in L_2$, then the following integral is bounded:

$$\int\limits_{t_0}^{t}\mathcal{M}_\kappa^2(s)\|\tilde{\kappa}(s)\|^2 ds \leqslant \frac{-V(t) + V(t_0) + \gamma\chi^{-1}\int_{t_0}^{t}\|\mathcal{W}_\kappa(s)\|^2 ds}{\gamma(1-\chi)} < \infty.$$

There exists the following lower bound for the integrand (here $\mathcal{M}_\kappa(t) > 0$ is assumed to hold almost everywhere and, consequently, $\operatorname{ess\,inf}\limits_t \mathcal{M}_\kappa^2(t) \neq 0$):

$$\operatorname*{ess\,inf}_t \mathcal{M}_\kappa^2(t)\|\tilde{\kappa}(t)\|^2 \leqslant \mathcal{M}_\kappa^2(t)\|\tilde{\kappa}(t)\|^2,$$

which, following Lemma A4, means that $\tilde{\kappa} \in L_2$.

Considering the error $\tilde{\psi}_b(t)$, the following is obtained:

$$\begin{aligned}\tilde{\psi}_b(t) &= \hat{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\hat{\theta}(t) - \rho\mathcal{L}_\psi\mathcal{L}_b\theta \pm \hat{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\theta \\ &= \hat{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t) + \tilde{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\theta \pm \rho\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t) \\ &= \tilde{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t) + \tilde{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\theta + \rho\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t),\end{aligned}$$

from which it is written that

$$\left.\begin{array}{l}\tilde{\theta} \in L_2 \cap L_\infty \\ \tilde{\rho} \in L_2 \cap L_\infty\end{array}\right\} \Rightarrow \tilde{\psi}_b \in L_2 \cap L_\infty, \left.\begin{array}{l}\lim\limits_{t\to\infty}\tilde{\theta}(t) = 0 \\ \lim\limits_{t\to\infty}\tilde{\rho}(t) = 0\end{array}\right\} \Rightarrow \lim_{t\to\infty}\tilde{\psi}_b(t) = 0.$$

Proof of Theorem 3. Owing to (3.11) and using (3.15), it is written for $W(t)$:

$$W(t) = \frac{1}{T}\varepsilon(t),$$

$$\varepsilon(t) = \int\limits_{\max(t_0, t-T)}^{t} \zeta_{iv}(s)\,w(s)\,ds,$$

from which, if Assumption 3 is met, it is obtained that $|W_i(t)| \leqslant \frac{1}{T}c$.

If C1 is satisfied, the following holds for the regressor $\Phi(t)$:

$$\|\Phi(t)\| = \frac{1}{T}\left\|\int\limits_{\max(t_0,t-T)}^{t}\zeta_{iv}(s)\,\varphi^\top(s)\,ds\right\| \leqslant \frac{1}{T}T\sup_t\left\|\zeta_{iv}(t)\,\varphi^\top(t)\right\| = \sup_t\left\|\zeta_{iv}(t)\,\varphi^\top(t)\right\|,$$

therefore, there exists a scalar $c_{\mathcal{W}} > 0$ such that

$$\|\mathcal{W}(t)\| \leqslant \frac{1}{T}c_{\mathcal{W}},\ \|\mathcal{W}_\rho(t)\| \leqslant \frac{1}{T}c_{\mathcal{W}}. \tag{B.7}$$

Based on (B.7), the errors $\tilde{\theta}(t),\ \tilde{\rho}(t),\ \tilde{\psi}_b(t)$ are defined as:

$$\begin{aligned}\tilde{\theta}(t) &= \phi_{\tilde{\theta}}(t, t_0)\,\tilde{\theta}(t_0) + \int\limits_{t_0}^{t}\phi_{\tilde{\theta}}(t,\tau)\,\Delta(\tau)\,\mathcal{W}(\tau)\,d\tau, \\ \tilde{\rho}(t) &= \phi_{\tilde{\rho}}(t,\ t_0)\,\tilde{\theta}(t_0) + \int\limits_{t_0}^{t}\phi_{\tilde{\rho}}(t,\tau)\,\mathcal{M}_\rho(\tau)\,\mathcal{W}_\rho(\tau)\,d\tau, \\ \tilde{\psi}_b(t) &= \tilde{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t) + \tilde{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\theta + \rho\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t),\end{aligned} \tag{B.8}$$

where $\phi_{\tilde{\theta}}(t,\tau) = e^{-\gamma\int\limits_{\tau}^{t}\Delta^2(s)ds}$, $\phi_{\tilde{\rho}}(t,\tau) = e^{-\gamma\int\limits_{\tau}^{t}\Delta^2(s)ds}$.

Owing to (B.7), (3.13), if C1 is met, then it holds that $\Delta\mathcal{W},\ \mathcal{M}_\rho\mathcal{W}_\rho \in L_\infty$, and if C2 is satisfied, then $\phi_{\tilde{\theta}},\ \phi_{\tilde{\rho}} \in L_1$, from which, using (B.8) and owing to (B.7), it immediately follows that there exist $\delta_0 > 0$ and $\delta_1 \in L_1,\ \lim\limits_{t\to\infty}\delta_1(t) = 0$ such that the inequality (3.16) is satisfied.

Proof of Theorem 4. Having substituted (2.2) + (3.17) into (2.1) and subtracted (2.7) from the obtained result, it is written:

$$\begin{aligned}\tilde{y}(t) &= W_{cl}(\theta_{cl},\ s)\left[\mathrm{sat}_{f_{\max}}\left\{-\hat{f}(t) \pm f_f(t) \pm f(t)\right\} + f(t)\right] \\ &= W_{cl}(\theta_{cl},\ s)\left[\mathrm{sat}_{f_{\max}}\left\{\tilde{f}_f(t) - \tilde{f}(t) - f(t)\right\} + f(t)\right],\end{aligned} \tag{B.9}$$

where

$$\begin{aligned}
\tilde{f}_f(t) &= -\hat{f}(t) + f_f(t) \\
&= -\psi_a^\top(\Theta)\hat{h}_\varepsilon(t) - \left(\hat{\psi}_a^\top(t) - \psi_a^\top(\Theta)\right)\hat{h}_y(t) \pm \hat{\psi}_a^\top(t)h_y(t) \\
&\quad + \psi_a^\top(\Theta)h_\varepsilon(t) + \left(\psi_a^\top(\theta_a) - \psi_a^\top(\Theta)\right)h_y(t) \\
&= -\psi_a^\top(\Theta)\tilde{h}_\varepsilon(t) - \tilde{\psi}_a^\top(t)h_y(t) - \left(\hat{\psi}_a^\top(t) - \psi_a^\top(\Theta)\right)\tilde{h}_y(t) \\
&= -\psi_a^\top(\Theta)\tilde{h}_\varepsilon(t) - \tilde{\psi}_a^\top(t)h_y(t) - \left(\psi_a^\top(\theta_a) - \psi_a^\top(\Theta)\right)\tilde{h}_y(t) - \tilde{\psi}_a^\top(t)\tilde{h}_y(t),
\end{aligned} \tag{B.10}$$

and (owing to Lemma B1)

$$\tilde{h}_y(t) = \begin{bmatrix} \mathcal{H}(t,s)[\tilde{y}_f(t)] \\ \vdots \\ \mathcal{H}(t,s)\left[s^{n-1}[\tilde{y}_f(t)]\right] \\ \mathcal{H}(t,s)[s^n[\tilde{y}_f(t)]] \end{bmatrix}, \quad \tilde{h}_\varepsilon(t) = \begin{bmatrix} \mathcal{H}(t,s)[\tilde{\varepsilon}_f(t)] \\ \vdots \\ \mathcal{H}(t,s)\left[s^{n-1}[\tilde{\varepsilon}_f(t)]\right] \\ \mathcal{H}(t,s)[s^n[\tilde{\varepsilon}_f(t)]] \end{bmatrix}. \tag{B.11}$$

As (B.9)–(B.11) depend from the errors $\tilde{\varepsilon}_f(t) = \hat{\varepsilon}_f(t) - \varepsilon_f(t)$ and $\tilde{y}_f(t) = \hat{y}_f(t) - y_f(t)$, then, using (3.5a)–(3.5c) and (3.18b), the differential equations for such errors are obtained:

$$\begin{aligned}
&\Sigma_1\colon \begin{cases} \dot{\tilde{\xi}}_y = \left(A_b - \tilde{\psi}_b e_1^\top\right)\tilde{\xi}_y - \tilde{\psi}_b y_f + \tilde{\rho} e_m y \\ \tilde{y}_f = \begin{cases} e_1^\top\tilde{\xi}_y, \text{ if } m \geqslant 1 \\ \tilde{\rho} y, \text{ if } m = 0, \end{cases} \end{cases} \\
&\Sigma_2\colon \begin{cases} \dot{\tilde{\xi}}_\varepsilon = \left(A_b - \tilde{\psi}_b e_1^\top\right)\tilde{\xi}_\varepsilon - \tilde{\psi}_b\varepsilon_f + (\rho + \tilde{\rho})e_m\tilde{\varepsilon} + \tilde{\rho}e_m\varepsilon^* \\ \tilde{\varepsilon}_f = \begin{cases} e_1^\top\tilde{\xi}_\varepsilon, \text{ if } m \geqslant 1 \\ \tilde{\rho}\tilde{\varepsilon} + \rho\tilde{\varepsilon} + \tilde{\rho}\varepsilon^*, \text{ if } m = 0. \end{cases} \end{cases}
\end{aligned} \tag{B.12}$$

The system Σ_3 depends form the error $\tilde{\varepsilon}(t) = \varepsilon(t) - \varepsilon^*(t) = y(t) - \hat{y}(t) - y(t) + \hat{y}^*(t)$, thus the differential equation for it is obtained as:

$$\Sigma_3\colon \begin{cases} \dot{\tilde{x}}(t) = \left(A_0 - \Theta e_1^\top\right)\tilde{x}(t) + \tilde{\theta}_b(t)u(t) \\ \tilde{\varepsilon}(t) = e_1^\top\tilde{x}(t). \end{cases} \tag{B.13}$$

Further proof of this theorem is based on Lemma A5. In order to apply it, first of all, we need to show that $\tilde{f}_f \in L_2 \cap L_\infty$ and $\lim\limits_{t\to\infty}\tilde{f}_f(t) = 0$. To that end, equation (B.13) is rewritten in the operator form as

$$\tilde{\varepsilon}(t) = e_1^\top\left(sI - A_0 + \Theta e_1^\top\right)\left[\tilde{\theta}_b(t)u(t)\right],$$

and, as $\tilde{\theta}_b \in L_2 \cap L_\infty$ (see Theorem 2) and $u \in L_\infty$ (owing to Assumption 1 and equation (3.17)), then, using Lemma A1, it holds that $\tilde{\varepsilon} \in L_2 \cap L_\infty$.

Further proof differs for cases $m = 0$ and $1 \leqslant m \leqslant n - 1$.

A) If $m = 0$, then we have

$$\left\{ \begin{matrix} \tilde{\rho},\ \tilde{\varepsilon} \in L_2 \cap L_\infty \\ y,\ \varepsilon^* \in L_\infty \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} \tilde{\varepsilon}_f \in L_2 \cap L_\infty \\ \tilde{y}_f \in L_2 \cap L_\infty \end{matrix} \right\}.$$

B) As for $1 \leqslant m \leqslant n - 1$, the following quadratic form is introduced:

$$V = \tilde{\xi}_\varepsilon^\top P\tilde{\xi}_\varepsilon, \tag{B.14}$$

where $A_b^\top P + PA_b = -Q,\ Q = Q^\top > 0$.

Considering the system Σ_2, the derivative of (B.14) is written as:

$$\begin{aligned} \dot{V} &= \tilde{\xi}_\varepsilon^\top \left(P\left[A_b - \tilde{\psi}_b e_1^\top \right] + \left[A_b - \tilde{\psi}_b e_1^\top \right]^\top P \right) \tilde{\xi}_\varepsilon + 2\tilde{\xi}_\varepsilon^\top P \left(-\tilde{\psi}_b \varepsilon_f + (\rho + \tilde{\rho})\, e_m \tilde{\varepsilon} + \tilde{\rho} e_m \varepsilon^* \right) \\ &= -\tilde{\xi}_\varepsilon^\top \left(Q + P\tilde{\psi}_b e_1^\top + e_1 \tilde{\psi}_b^\top P \right) \tilde{\xi}_\varepsilon + 2\tilde{\xi}_\varepsilon^\top P \left(-\tilde{\psi}_b \varepsilon_f + (\rho + \tilde{\rho})\, e_n \tilde{\varepsilon} + \tilde{\rho} e_n \varepsilon^* \right) \\ &\leqslant -\left(\lambda_{\min}(Q) - 2\lambda_{\max}(P) \left\| \tilde{\psi}_b \right\| - \chi_1^{-1} \lambda_{\max}^2 (P) \right) \left\| \tilde{\xi}_\varepsilon \right\|^2 + \epsilon, \end{aligned} \tag{B.15}$$

where $\chi_1 > 0$ and $\epsilon = \chi_1 \left(\left\| \tilde{\psi}_b \varepsilon_f \right\| + \left\| (\rho + \tilde{\rho})\, e_n \tilde{\varepsilon} \right\| + \left\| \tilde{\rho} e_n \varepsilon^* \right\| \right)^2$.

For all Q and P there exist scalars $\chi_1 > 0$ and $\chi_2 > 0$ such that

$$\lambda_{\min}(Q) - \chi_1^{-1} \lambda_{\max}^2 (P) \geqslant \chi_2 > 0.$$

This fact allows one to rewrite (B.15) as:

$$\dot{V} \leqslant -\left(\chi_2 - 2\lambda_{\max}(P) \left\| \tilde{\psi}_b \right\| \right) \left\| \tilde{\xi}_\varepsilon \right\|^2 + \epsilon. \tag{B.16}$$

As $\tilde{\psi}_b \in L_\infty$ and $\tilde{\psi}_b(t) \to 0$ when $t \to \infty$ (see Theorem 2), then there always exist scalars $\sigma_1 > 0$ and $\sigma_2 > 0$ and time instant $\infty > t_V \geqslant t_0$ such that

$$0 < \frac{-\left(\chi_2 - 2\lambda_{\max}(P) \left\| \tilde{\psi}_b \right\| \right)}{\lambda_{\max}(P)} \leqslant \sigma_1, \text{ for all } t \leqslant t_V,$$

$$\frac{\chi_2 - 2\lambda_{\max}(P) \left\| \tilde{\psi}_b \right\|}{\lambda_{\max}(P)} \geqslant \sigma_2 > 0, \text{ for all } t \geqslant t_V.$$

Using it, equation (B.16) is rewritten as follows:

$$\dot{V}(t) \leqslant \begin{cases} \sigma_1 V(t) + \epsilon(t), \text{ for all } t \leqslant t_V \\ -\sigma_2 V(t) + \epsilon(t), \text{ for all } t > t_V. \end{cases} \tag{B.17}$$

As $\tilde{\psi}_b,\ \tilde{\rho},\ \tilde{\varepsilon} \in L_2 \cap L_\infty$ (see Theorem 2 and above-given proof) and $\varepsilon_f \in L_\infty$, $\varepsilon^* \in L_\infty$ (owing to Assumption 1 and equation (3.17)), then $\epsilon \in L_1 \cap L_\infty$. As $\infty > t_V \geqslant t_0$ and $\epsilon \in L_1 \cap L_\infty$, then $V(t)$ is bounded for all $t \in [t_0,\ t_V]$. Given $\epsilon \in L_1 \cap L_\infty$, then, using Lemma A1, the solution of (B.17) is also bounded for all $t \geqslant t_V$ and, consequently, $V \in L_\infty$.

Having integrated (B.17), it is obtained:

$$V(t) \leqslant V(t_V) - \sigma_2 \int\limits_{t_V}^{t} V(s)\, ds + \int\limits_{t_V}^{t} \epsilon(s)\, ds. \tag{B.18}$$

As $V \in L_\infty$ and $\epsilon \in L_1$, then the following integral is bounded:

$$\sigma_2 \int\limits_{t_V}^{t} V(s)\, ds \leqslant -V(t) + V(t_V) + \int\limits_{t_V}^{t} \epsilon(s)\, ds < \infty,$$

from which it holds that $\tilde{\varepsilon}_f \in L_2 \cap L_\infty$.

Having repeated the above reasoning (B.14)–(B.18), it is obtained that $\tilde{y}_f \in L_2 \cap L_\infty$. Therefore, using Lemma A1, for all $0 \leqslant m \leqslant n-1$ in the sense of the following implication

$$\left.\begin{array}{l}\tilde{\varepsilon}_f \in L_2 \cap L_\infty \\ \tilde{y}_f \in L_2 \cap L_\infty\end{array}\right\} \Rightarrow \left.\begin{array}{l}s^i\left[\tilde{\varepsilon}_f(t)\right] \in L_2 \cap L_\infty \\ s^i\left[\tilde{y}_f(t)\right] \in L_2 \cap L_\infty\end{array}\right\} \forall i=0,\ldots,n \tag{B.19}$$

we almost always have $\tilde{h}_\varepsilon \in L_2 \cap L_\infty$, $\tilde{h}_y \in L_2 \cap L_\infty$ (for example, if $\frac{d}{dt}[\mu(t)] = 0$, then $\mathcal{H}(t,s)\left[s^i[.]\right] = \mathcal{H}(t,s)\, s^i[.]$ and $\tilde{h}_\varepsilon \in L_2 \cap L_\infty$, $\tilde{h}_y \in L_2 \cap L_\infty$ follow directly from Lemma A1).

Taking into consideration $\tilde{\psi}_a \in L_2 \cap L_\infty$ (see Theorem 2), on the basis of equation (B.10) it is finally obtained that $\tilde{f}_f \in L_2 \cap L_\infty$. As $\tilde{f}_f \in L_2 \cap L_\infty$, $\tilde{f} \in L_p \cap L_\infty$ for $p \in [1,\ \infty)$ (see Proposition 1), then there exists a sufficiently large scalar $f_{\max}$ such that the following holds

$$\operatorname{sat}_{f_{\max}}\left\{\tilde{f}_f(t) - \tilde{f}(t) - f(t)\right\} = \tilde{f}_f(t) - \tilde{f}(t) - f(t),$$

thus, equation (B.9) is rewritten as:

$$\tilde{y}(t) = W_{cl}(\theta_{cl},\ s)\left[\tilde{f}_f(t) - \tilde{f}(t)\right],$$

from which, owing to $\tilde{f}_f \in L_2 \cap L_\infty$, $\tilde{f} \in L_p \cap L_\infty$ and using Lemma A5, it is obtained that $\lim\limits_{t\to\infty} \tilde{y}(t) = 0$.

Proof of Theorem 5. In accordance with proof of Proposition 1 and Theorem 4, when $\mu(t) = \mu > 0$, the closed-loop system is described by the following equations:

$$\begin{cases}\dot{\eta}(t) = \mu G \eta(t) + e_{n+1} m_0 \mu \lambda(t) \\ \tilde{f}(t) = e_1^\top \eta(t),\end{cases} \tag{B.20a}$$

$$\tilde{y}(t) = W_{cl}(\theta_{cl},\ s)\left[\operatorname{sat}_{f_{\max}}\left\{\tilde{f}_f(t) - \tilde{f}(t) - f(t)\right\} + f(t)\right], \tag{B.20b}$$

$$\tilde{f}_f(t) = -\psi_a^\top(\Theta)\,\tilde{h}_\varepsilon(t) - \tilde{\psi}_a^\top(t)\, h_y(t) - \left(\psi_a^\top(\theta_a) - \psi_a^\top(\Theta)\right)\tilde{h}_y(t) - \tilde{\psi}_a^\top(t)\,\tilde{h}_y(t), \tag{B.20c}$$

where $\tilde{f} \in L_\infty$ as $\mu\lambda \in L_\infty$ and G is a Hurwitz matrix, and, if Assumption 1 is met, repeating proof (B.12)–(B.19) from Theorem 4, we have for $\tilde{f}_f(t)$ that

$$\tilde{\rho},\ \tilde{\theta},\ \tilde{\psi}_b \in L_2 \cap L_\infty \Rightarrow \tilde{f}_f \in L_2 \cap L_\infty.$$

Then there exists a sufficiently large scalar $f_{\max}$ such that the following holds:

$$\tilde{y}(t) = \underbrace{W_{cl}(\theta_{cl},\ s)\left[\tilde{f}_f(t)\right]}_{\tilde{y}_1(t)} - \underbrace{W_{cl}(\theta_{cl},\ s)\left[\tilde{f}(t)\right]}_{\tilde{y}_2(t)},$$

and, owing to $\tilde{f}_f \in L_2 \cap L_\infty$ and using Lemma A5, it is obtained that $\lim\limits_{t\to\infty} \tilde{y}_1(t) = 0$.

In order to complete the proof, equation (B.20a) is considered together with the system

$$\begin{aligned}\dot{z}(t) &= A_{cl} z(t) + e_{n_{cl}} \tilde{f}(t), \\ \tilde{y}_2(t) &= \left[0_{1\times(n_{cl}-(m_{cl}+1))}\ \ \theta_{b.cl}^\top\right] z(t),\end{aligned} \tag{B.21}$$

where

$$A_{cl} = A_0 + e_{n_{cl}} \theta_{a.cl}^\top,\ A_0 = \begin{bmatrix} 0_{n_{cl}} & \begin{matrix} I_{n_{cl}-1} \\ 0_{1\times(n_{cl}-1)} \end{matrix} \end{bmatrix},\ e_{n_{cl}} = \begin{bmatrix} 0_{n_{cl}-1} \\ 1 \end{bmatrix},$$

and the parameters $\theta_{a.cl},\ \theta_{b.cl}$ are equivalents of $\theta_a,\ \theta_b$ and match the parameters of the numerator and denominator of the transfer function of the closed-loop system (2.4).

The following quadratic form is introduced to analyze stability of (B.20a) + (B.21):

$$V = z^\top P_z z + \frac{1}{\mu}\eta^\top P_\eta \eta, \tag{B.22}$$

where $P_z A_{cl} + A_{cl}^\top P_z = -Q_z,\ P_\eta G + G^\top P_\eta = -Q_\eta$ and matrices $Q_z = Q_z^\top > 0,\ Q_\eta = Q_\eta^\top > 0$ are chosen such that for some scalars $\chi_z \in (0,1),\ \chi_\eta \in (0,1)$ it holds that:

$$\begin{aligned} &0 < c_z \leqslant \lambda_{\min}\left(Q_z - \chi_z^{-1} P_z e_{n_{cl}} e_{n_{cl}}^\top P_z\right), \\ &0 < c_\eta \leqslant \lambda_{\min}\left(Q_\eta - m_0 \chi_\eta^{-1} P_\eta e_{n+1} e_{n+1}^\top P_\eta - \chi_z e_1 e_1^\top\right). \end{aligned}$$

Owing to equations (B.20a) and (B.21), the derivative of (B.22) is written as:

$$\begin{aligned} \dot{V} = {} & z^\top \left[P_z A_{cl} + A_{cl}^\top P_z\right] z + 2 z^\top P_z e_{n_{cl}} e_1^\top \eta + \eta^\top \left[P_\eta G + G^\top P_\eta\right] \eta + 2\eta^\top P_\eta e_{n+1} m_0 \lambda \\ = {} & -z^\top Q_z z - \eta^\top Q_\eta \eta + 2 z^\top P_z e_{n_{cl}} e_1^\top \eta + 2\eta^\top P_\eta e_{n+1} m_0 \lambda. \end{aligned} \tag{B.23}$$

Having applied the inequalities:

$$\begin{aligned} &2 z^\top P_z e_{n_{cl}} e_1^\top \eta \leqslant \chi_z^{-1} z^\top P_z e_{n_{cl}} e_{n_{cl}}^\top P_z z + \chi_z \eta^\top e_1 e_1^\top \eta, \\ &2\eta^\top P_\eta e_{n+1} \lambda \leqslant \chi_\eta^{-1} \eta^\top P_\eta e_{n+1} e_{n+1}^\top P_\eta \eta + \chi_\eta \lambda^2, \end{aligned}$$

it is obtained that:

$$\begin{aligned} \dot{V} = & -z^\top \left[Q_z - \chi_z^{-1} P_z e_{n_{cl}} e_{n_{cl}}^\top P_z\right] z \\ & -\eta^\top \left[Q_\eta - m_0 \chi_\eta^{-1} P_\eta e_{n+1} e_{n+1}^\top P_\eta - \chi_z e_1 e_1^\top\right] \eta + \chi_\eta m_0 \lambda^2 \\ \leqslant & -\min\left\{\frac{c_z}{\lambda_{\max}(P_z)}, \frac{\mu c_\eta}{\lambda_{\max}(P_\eta)}\right\} V + \chi_\eta m_0 \lambda^2. \end{aligned} \tag{B.24}$$

As, owing to Assumption 2, it holds that $\lim\limits_{\mu\to\infty} \lambda^2(\mu) = 0$, then we have from (B.24) that $\lim\limits_{t\to\infty} \tilde{y}_1(t) = 0$ and $\lim\limits_{t\to\infty} |\tilde{y}(t)| \leqslant \epsilon$ for arbitrarily small scalar $\epsilon > 0$.

Proof of Theorem 6. In case $\mu(t) = \mu > 0$, owing to Theorem 5, the closed-loop control system is described by equations (B.20a)–(B.20c) and (B.21). Moreover, considering forced motion component $\tilde{y}_2(t)$ and repeating analysis (B.21)–(B.24), we have $\lim\limits_{t\to\infty} |\tilde{y}_2(t)| \leqslant \epsilon_2$ for arbitrarily small scalar $\epsilon_2 > 0$. Using the upper bound (3.16) and equations (B.10)–(B.13), the following upper bound for $\tilde{f}_f(t)$ is obtained via simple but tedious reasoning:

$$\left|\tilde{f}_f(t)\right| \leqslant \tilde{f}_{1f}(t) + T^{-1}\tilde{f}_{0f}, \tag{B.25}$$

where $\tilde{f}_{1f} \in L_2$ and $0 < \tilde{f}_{0f} < \infty$.

Then, in order to complete proof, the following system remains to be considered

$$\begin{aligned} &\dot{z}(t) = A_{cl} z(t) + e_{n_{cl}} \tilde{f}_f(t), \\ &\tilde{y}_1(t) = \begin{bmatrix} 0_{1\times(n_{cl}-(m_{cl}+1))} & \theta_{b.cl}^\top \end{bmatrix} z(t). \end{aligned} \tag{B.26}$$

The below-given quadratic form is introduced to analyze stability of (B.26):

$$V = z^{\top} P_z z, \tag{B.27}$$

where $P_z A_{cl} + A_{cl}^{\top} P_z = -Q_z$ and the matrix $Q_z = Q_z^{\top} > 0$ is chosen such that for some scalar $\chi_z \in (0,\ 1)$ it holds that $0 < c_z \leqslant \lambda_{\min}(Q_z) - \chi_z^{-1} \|P_z e_{n_{cl}}\|^2$.

Owing to (B.26), the derivative of (B.27) is written as:

$$\dot{V} = z^{\top} \left[P_z A_{cl} + A_{cl}^{\top} P_z \right] z + 2 z^{\top} P_z e_{n_{cl}} \tilde{f}_f. \tag{B.28}$$

Using (B.25), the following is obtained:

$$\begin{aligned}
&2 \left\| z^{\top} P_z e_{n_{cl}} \tilde{f}_f \right\| \leqslant 2 \|z\| \|P_z e_{n_{cl}}\| \left| \tilde{f}_f \right|, \\
&2 \|z\| \|P_z e_{n_{cl}}\| \left| \tilde{f}_f \right| \leqslant \chi_z^{-1} \|P_z e_{n_{cl}}\|^2 \|z\|^2 + \chi_z \tilde{f}_f^2, \\
&\tilde{f}_f^2 \leqslant \tilde{f}_{1f}^2(t) + 2T^{-1} \tilde{f}_{1f} \tilde{f}_{0f} + T^{-2} \tilde{f}_{0f}^2, \\
&2T^{-1} \tilde{f}_{1f} \tilde{f}_{0f} \leqslant \tilde{f}_{1f}^2(t) + T^{-2} \tilde{f}_{0f}^2,
\end{aligned}$$

and therefore, we have:

$$\dot{V} \leqslant -c_z \|z\|^2 + 2\chi_z \tilde{f}_{1f}^2(t) + 2T^{-2} \chi_z \tilde{f}_{0f}^2. \tag{B.29}$$

As $\tilde{f}_{1f} \in L_2$ and $\tilde{f}_{0f} < \infty$, then from (B.29) it holds that $\lim\limits_{t\to\infty} |\tilde{y}_1(t)| \leqslant \epsilon_1$ for arbitrarily small scalar $\epsilon_1 > 0$, which allows one to conclude that $\lim\limits_{t\to\infty} |\tilde{y}(t)| \leqslant \epsilon$ for arbitrarily small scalar $\epsilon > 0$.

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